1. Calcule as seguintes integrais:

   (a) **(1,5)** $\int \ln(x^2+4x+5)\mathrm{d}x$

   (b) **(1,5)** $\int \dfrac{x^5}{\sqrt{(4-x^2)^3}}\mathrm{d}x$

(a) $\int \ln(x^2+4x+5)\,dx = x\ln(x^2+4x+5) - \int \frac{x(2x+4)}{x^2+4x+5}dx$ integração por partes

$u = \ln(x^2+4x+5)$

$du = \frac{2x+4}{x^2+4x+5}dx$

$dv = dx \qquad v = x$

Temos então que calcular $2\int \frac{x^2+2x}{x^2+4x+5}dx$, Como o grau do numerador é igual ao grau do denominador, efetuamos a divisão:

$$\begin{array}{r|l} x^2+2x & x^2+4x+5 \\ \hline -x^2-4x & 1 \\ -2x-5 & \end{array}$$

Logo $\int \frac{x^2+2x}{x^2+4x+5}dx = \int dx + \int \frac{-2x-5}{x^2+4x+5}dx$,

$= x - \int \frac{2x+5}{x^2+4x+5}dx$.

Calcular então $\int \frac{2x+5}{x^2+4x+5}dx = \int \frac{2x+5\,dx}{(x+2)^2+1} = \int \frac{(2y-4+5)\,dy}{y^2+1}$

$y = x+2$

$dy = dx$

$x = y-2$

$= \int \frac{2y\,dy}{y^2+1} + \int \frac{dy}{y^2+1}$

$= \ln(y^2+1) + \operatorname{arctg} y + C = \ln(x^2+4x+5) + \operatorname{arctg}(x+2) + C$

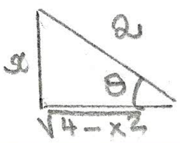
Portanto:

$$\boxed{\int \ln(x^2+4x+5)\,dx = x\ln(x^2+4x+5) + 2\left[x - \ln(x^2+4x+5) - \operatorname{arctg}(x+2)\right] + k \quad k \in \mathbb{R}}$$

(b) $\displaystyle\int \frac{x^5\,dx}{\sqrt{(4-x^2)^3}} \overset{(*)}{=} \int \frac{2^5 \operatorname{sen}^5\theta \; 2\cos\theta \; d\theta}{2^3 \cos^3\theta}$

(*) Substituição Trigonométrica

$$\boxed{\begin{aligned} * \quad x &= 2\operatorname{sen}\theta, \quad \theta \in \left]-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right[\\ dx &= 2\cos\theta\, d\theta \\ \sqrt{4-x^2} &= 2\cos\theta \end{aligned}}$$



$$= 2^3 \int \frac{\operatorname{sen}^5\theta}{\cos^2\theta}\, d\theta = 2^3 \int \frac{(1-\cos^2\theta)^2}{\cos^2\theta} \operatorname{sen}\theta \, d\theta$$

$$= 2^3 \left[\int \frac{1}{\cos^2\theta} \operatorname{sen}\theta\, d\theta - 2\int \operatorname{sen}\theta\, d\theta + \int \cos^2\theta \operatorname{sen}\theta\, d\theta \right]$$

$$\underset{\boxed{\begin{aligned} u &= \cos\theta \\ du &= -\operatorname{sen}\theta\, d\theta \end{aligned}}}{=} -2^3 \left[\int \frac{du}{u^2} - 2\int du + \int u^2\, du \right]$$

$$= -2^3 \left[\frac{u^{-1}}{-1} - 2u + \frac{u^3}{3} \right] + C$$

$$= -2^3 \left[-\frac{1}{\cos\theta} - 2\cos\theta + \frac{\cos^3\theta}{3} \right] + C$$

$$= -2^3 \left[-\frac{2}{\sqrt{4-x^2}} - \frac{2\sqrt{4-x^2}}{2} + \frac{\left(\sqrt{4-x^2}\right)^3}{3 \cdot 8} \right] + C$$

Assim:

$$\boxed{\int \frac{x^5}{\sqrt{(4-x^2)^3}}\, dx = \frac{16}{\sqrt{4-x^2}} + 8\sqrt{4-x^2} + \frac{\left(\sqrt{4-x^2}\right)^3}{3} + C}$$

2. **(2,0)** Calcule a integral

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{6}} \frac{\sec^2(x)}{(\mathrm{tg}(x)-2)(\mathrm{tg}(x)-1)^2}\mathrm{d}x.$$

Fazendo a substituição $t = \mathrm{tg}\, x$, temos $dt = \sec^2 x\, dx$,

$x = -\pi/4 \Rightarrow t = -1$ e $x = \pi/6 \Rightarrow t = 1/\sqrt{3}$

Assim, a integral fica $\displaystyle\int_{-1}^{1/\sqrt{3}} \frac{1}{(t-2)(t-1)^2}\, dt$.

Agora, $\displaystyle\frac{1}{(t-2)(t-1)^2} = \frac{A}{t-2} + \frac{B}{t-1} + \frac{C}{(t-1)^2} = \frac{A(t-1)^2 + B(t-1)(t-2) + C(t-2)}{(t-2)(t-1)^2}$

Resolvendo o sistema, obtemos $A = 1$, $B = -1$ e $C = -1$.

Portanto,

$$\int_{-1}^{1/\sqrt{3}} \frac{1}{(t-2)(t-1)^2}\, dt = \int_{-1}^{1/\sqrt{3}} \left(\frac{1}{t-2} - \frac{1}{t-1} - \frac{1}{(t-1)^2}\right) dt = \left(\ln|t-2| - \ln|t-1| + \frac{1}{t-1}\right)\Big|_{-1}^{1/\sqrt{3}}$$

$$= \ln\left(2 - \frac{1}{\sqrt{3}}\right) - \ln\left(1 - \frac{1}{\sqrt{3}}\right) + \frac{1}{\frac{1}{\sqrt{3}} - 1} - \ln 3 + \ln 2 - \frac{1}{2}.$$

3. **(1,5)** Seja $F(x) = \int_0^{\sqrt{x}} (x + t^2)\mathrm{e}^{t^2}\mathrm{d}t$, $x > 0$. Determine $F'(x)$ e $F''(x)$.

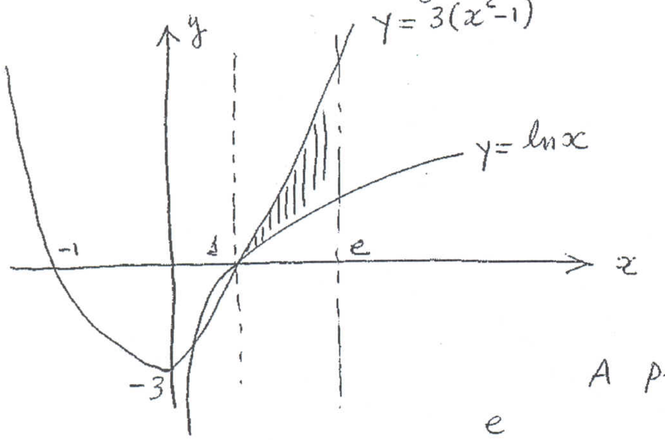
$F(x) = x\displaystyle\int_0^{\sqrt{x}} e^{t^2} dt + \int_0^{\sqrt{x}} t^2 e^{t^2} dt$. As funções e^{t^2} e $t^2 e^{t^2}$ são contínuas. Pelo TFC,

$$F'(x) = \int_0^{\sqrt{x}} e^{t^2} dt + x e^{x} \cdot \frac{1}{2\sqrt{x}} + x e^{x} \cdot \frac{1}{2\sqrt{x}} = \int_0^{\sqrt{x}} e^{t^2} dt + \sqrt{x}\, e^{x}$$

e

$$F''(x) = e^{x} \cdot \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{x}} e^{x} + \sqrt{x}\, e^{x} = e^{x}\left(\frac{1}{\sqrt{x}} + \sqrt{x}\right)$$

(a) Esboço da região R



(b) V_x volume do sólido obtido pela rotação de R em torno do eixo x

$$V_x = \pi \int_1^e f(x)^2 dx - \pi \int_1^e g(x)^2 dx$$

A primeira integral:

$$\pi \int_1^e (3(x^2-1))^2 dx = 9\pi \int_1^e (x^4 - 2x^2 + 1)\, dx = 9\pi \left[\frac{x^5}{5} - \frac{2x^3}{3} + x \right]_1^e =$$

$$= 9\pi \left[\left(\frac{e^5}{5} - \frac{2e^3}{3} + e \right) - \left(\frac{1}{5} - \frac{2}{3} + 1 \right) \right] = \frac{9\pi}{5} e^5 - 6\pi e^3 + 9\pi e - \frac{24}{5}\pi$$

A segunda integral será feita por partes:

$\ln x \xrightarrow{D} \frac{1}{x}$

$\ln x \xrightarrow{I} x\ln x - x$

$$\pi \int_1^e \ln^2 x\, dx = \pi \underbrace{\left[(\ln x)(x\ln x - x) \right]_1^e}_{\equiv 0} - \pi \int_1^e (\ln x - 1)\, dx$$

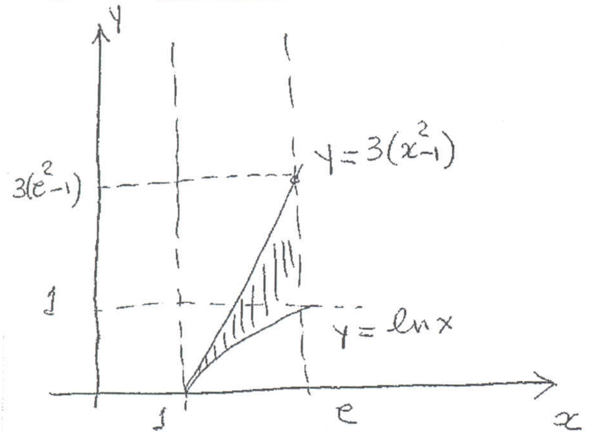
$$\pi \int_1^e \ln^2 x\, dx = -\pi \left[x\ln x - 2x \right]_1^e = -\pi \left[-e + 2 \right] = \pi e - 2\pi$$

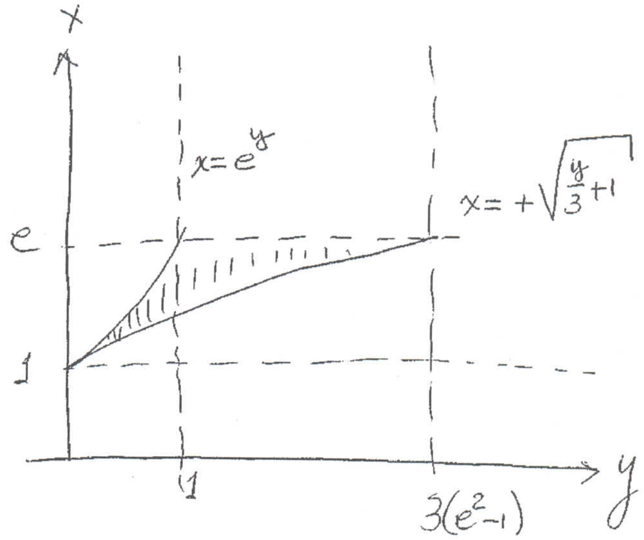
Assim, $V_x = \left(\frac{9\pi}{5} e^5 - 6\pi e^3 + 9\pi e - \frac{24}{5}\pi \right) - \left(\pi e - 2\pi \right)$

$$\boxed{V_x = \frac{9\pi}{5} e^5 - 6\pi e^3 + 8\pi e - \frac{12\pi}{5}}$$

(c) V_y = volume do sólido obtido pela rotação de R em torno do eixo y:

Primeira solução:





$$V_y = \pi\int_0^1 (e^y)^2\,dy - \pi\int_0^1 \left(\sqrt{\tfrac{y}{3}+1}\right)^2 dy + \pi e^2(3e^2-3-1) - \pi\int_1^{3(e^2-1)} \left(\sqrt{\tfrac{y}{3}+1}\right)^2 dy$$

$$V_y = \pi\left[\frac{e^{2y}}{2}\right]_0^1 - \pi\left[\frac{y^2}{6}+y\right]_0^1 + \pi e^2(3e^2-4) - \pi\left[\frac{y^2}{6}+y\right]_1^{3(e^2-1)}$$

$$V_y = \pi\left(\frac{e^2}{2}-\frac{1}{2}\right) - \pi\left(\frac{1}{6}+1\right) + \pi e^2(3e^2-4) - \pi\left[\frac{3(e^2-1)^2}{2} + 3(e^2-1) - \frac{1}{6} - 1\right]$$

$$\boxed{V_y = \frac{3\pi}{2}e^4 - \frac{7\pi}{2}e^2 + \pi}$$

Segunda solução: $V_y = 2\pi\int_1^e x\,3(x^2-1)\,dx - 2\pi\int_1^e x\ln x\,dx$

$$V_y = 2\pi\left[\frac{3x^4}{4} - \frac{3x^2}{2}\right]_1^e - 2\pi\int_1^e x\ln x\,dx$$

$$V_y = 2\pi\left[\left(\frac{3e^4}{4} - \frac{3e^2}{2}\right) - \left(\frac{3}{4} - \frac{3}{2}\right)\right] - 2\pi\int_1^e x\ln x\,dx$$

$$2\pi\int_1^e x\ln x\,dx = \left[\frac{x^2}{2}\ln x\right]_1^e \pi 2 - 2\pi\cdot\frac{1}{2}\int_1^e x\,dx$$

$$\ln x \xrightarrow{D} \frac{1}{x}$$

$$x \xrightarrow{I} \frac{x^2}{2}$$

$$2\pi\int_1^e x\ln x\,dx = 2\pi\left(\frac{e^2}{2}\right) - \pi\left[\frac{x^2}{2}\right]_1^e$$

$$= \pi e^2 - \pi\left(\frac{e^2}{2} - \frac{1}{2}\right) = \frac{\pi e^2}{2} + \frac{\pi}{2}$$

Assim,

$$V_y = 2\pi\left[\left(\frac{3e^4}{4} - \frac{3e^2}{2}\right) - \left(-\frac{3}{4}\right)\right] - \frac{\pi e^2}{2} - \frac{\pi}{2}$$

$$\boxed{V_y = \frac{3\pi}{2}e^4 - \frac{7\pi}{2}e^2 + \pi}$$

1. Calcule as seguintes integrais:

   (a) **(1,5)** $\int \ln(x^2 - 4x + 5)dx$

   (b) **(1,5)** $\int \dfrac{x^5}{\sqrt{(9-x^2)^3}}dx$

(a) $\int \underbrace{\ln(x^2-4x+5)}_{u}\underbrace{dx}_{dv} = x\ln(x^2-4x+5) - \int \dfrac{x(2x-4)}{x^2-4x+5}dx$

Utilizar integração por partes:

$u = \ln(x^2-4x+5)$

$du = \dfrac{2x-4}{x^2-4x+5}dx$

$dv = dx$

$v = x$

$= x\ln(x^2-4x+5) - 2\int \dfrac{x^2-2x}{x^2-4x+5}dx$

Calcular $\int \dfrac{x^2-2x}{x^2-4x+5}dx \overset{(*)}{=} \int dx + \int \dfrac{2x-5}{x^2-4x+5}dx$

(*) Como o grau do numerador é igual ao grau do denominador, temos que efetuar a divisão

$$\begin{array}{r|l} x^2 - 2x & x^2-4x+5 \\ \hline -x^2+4x-5 & 1 \\ \hline 2x-5 & \end{array}$$

Calcular $\int \dfrac{2x-5}{x^2-4x+5}dx = \int \dfrac{2x-5}{(x-2)^2+1}dx = \int \dfrac{2(y+2)-5}{y^2+1}dy$

$y = x-2$

$dy = dx$

$x = y+2$

$= \int \dfrac{2y}{y^2+1}dy - \int \dfrac{dy}{y^2+1} = \ln(y^2+1) - \operatorname{arctg} y + C$

$= \ln(x^2-4x+5) - \operatorname{arctg}(x-2) + C'$

Assim

$$\int \ln(x^2-2x+5)dx = x\ln(x^2-4x+5) - 2\left[x + \ln(x^2-4x+5) - \operatorname{arctg}(x-2)\right] + k$$

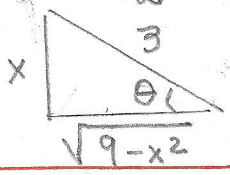
(b) $\displaystyle\int \frac{x^5}{\sqrt{(9-x^2)^3}}\,dx \overset{(*)}{=} \int \frac{3^5\,\text{sen}^5\theta\ 3\cos\theta\,d\theta}{3^3\cos^3\theta}$

(*) Substituição Trigonométrica

$x = 3\,\text{sen}\,\theta \ , \qquad -\pi/2 < \theta < \pi/2$

$dx = 3\cos\theta\,d\theta$

$\sqrt{9-x^2} = 3\cos\theta$



$$= 3^3 \int \frac{\text{sen}^5\theta}{\cos^2\theta}\,d\theta = 3^3 \int \frac{(1-\cos^2\theta)^2}{\cos^2\theta}\,\text{sen}\,\theta\,d\theta$$

$$= -3^3 \int \frac{(1-u^2)^2}{u^2}\,du = -3^3 \int \left(\frac{1}{u^2} - 2 + u^2\right) du$$

$u = \cos\theta$

$du = -\text{sen}\,\theta\,d\theta$

$$= -3^3\left[-\frac{1}{u} - 2u + \frac{u^3}{3}\right] + C$$

$$= -3^3\left[-\frac{1}{\cos\theta} - 2\cos\theta + \frac{\cos^3\theta}{3}\right] + C$$

$$\boxed{= -3^3\left[-\frac{3}{\sqrt{9-x^2}} - \frac{2\sqrt{9-x^2}}{3} + \frac{(\sqrt{9-x^2})^3}{3^4}\right] + C}$$

2. **(2,0)** Calcule a integral

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{6}} \frac{\sec^2(x)}{(\text{tg}(x)-1)(\text{tg}(x)-2)^2}dx.$$

Fazendo a substituição $t = \text{tg}\,x$, temos $dt = \sec^2 x\,dx$,

$x = -\pi/3 \Rightarrow t = -\sqrt{3}$ e $x = \pi/6 \Rightarrow t = 1/\sqrt{3}$.

Assim, a integral fica $\displaystyle\int_{-\sqrt{3}}^{1/\sqrt{3}} \frac{1}{(t-1)(t-2)^2}dt$

Agora, $\displaystyle\frac{1}{(t-1)(t-2)^2} = \frac{A}{t-1} + \frac{B}{t-2} + \frac{C}{(t-2)^2} = \frac{A(t-2)^2 + B(t-1)(t-2) + C(t-1)}{(t-1)(t-2)^2}$

Resolvendo o sistema, obtemos $A = 1$, $B = -1$ e $C = 1$.

Portanto,

$$\int_{-\sqrt{3}}^{1/\sqrt{3}} \frac{1}{(t-1)(t-2)^2}dt = \int_{-\sqrt{3}}^{1/\sqrt{3}} \left(\frac{1}{t-1} - \frac{1}{t-2} + \frac{1}{(t-2)^2}\right)dt = \left(\ln|t-1| - \ln|t-2| - \frac{1}{t-2}\right)\Big|_{-\sqrt{3}}^{1/\sqrt{3}}$$

$$= \ln\left(1-\frac{1}{\sqrt{3}}\right) - \ln\left(2-\frac{1}{\sqrt{3}}\right) - \frac{1}{\frac{1}{\sqrt{3}}-2} - \ln(1+\sqrt{3}) + \ln(2+\sqrt{3}) - \frac{1}{\sqrt{3}+2}$$

3. **(1,5)** Seja $F(x) = \int_0^{\sqrt{x}}(x+t^2)e^{t^2}dt$, $x > 0$. Determine $F'(x)$ e $F''(x)$.

$F(x) = x\displaystyle\int_0^{\sqrt{x}} e^{t^2}dt + \int_0^{\sqrt{x}} t^2 e^{t^2}dt$. As funções e^{t^2} e $t^2e^{t^2}$ são contínuas. Pelo TFC,

$$F'(x) = \int_0^{\sqrt{x}} e^{t^2}dt + x.e^{x}.\frac{1}{2\sqrt{x}} + xe^{x}.\frac{1}{2\sqrt{x}} = \int_0^{\sqrt{x}} e^{t^2}dt + \sqrt{x}e^{x}$$

e

$$F''(x) = e^{x}.\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{x}}e^{x} + \sqrt{x}e^{x} = e^{x}\left(\frac{1}{\sqrt{x}} + \sqrt{x}\right)$$