Questão 1 (Turma A)

Item a) Aplicamos duas vezes a integração por partes:

$$\begin{aligned}\int x^2(\ln(x))^2dx =& \frac{x^3}{3}(\ln(x))^2 - \frac{2}{3}\int x^2\ln(x)dx \\ =& \frac{x^3}{3}(\ln(x))^2 - \frac{2}{3}(\frac{x^3}{3}\ln(x) - \frac{1}{3}\int x^2 dx) \\ =& \frac{x^3}{3}(\ln(x))^2 - \frac{2}{9}x^3\ln(x) + \frac{2}{27}x^3 + c\end{aligned}$$

Item b) Usamos a substituição $u = x^3$ e, em seguida, $v = \tan(u)$:

$$\begin{aligned}\int x^2\tan^3(x^3)\sec^4(x^3)dx =& \frac{1}{3}\int \tan^3(u)\sec^4(u)du \\ =& \frac{1}{3}\int \tan^3(u)(1+\tan^2(u))\sec^2(u)du \\ =& \frac{1}{3}\int v^3(1+v^2)dv \\ =& \frac{1}{3}\left(\frac{v^4}{4} + \frac{v^6}{6}\right) + c \\ =& \frac{\tan^4(x^3)}{12} + \frac{\tan^6(x^3)}{18} + c\end{aligned}$$

Item c) Usamos a substituição trigonométrica $x - 2 = \tan(\theta)$:

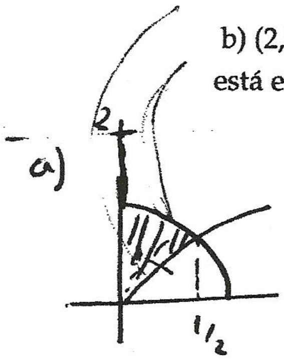
$$\begin{aligned}\int \frac{x+1}{(x^2-4x+5)^2}dx =& \int \frac{x+1}{((x-2)^2+1)^2}dx \\ =& \int \frac{\tan(\theta)+3}{(\tan^2(\theta)+1)^2}\sec^2(\theta)d\theta \\ =& \int \frac{\tan(\theta)+3}{\sec^2(\theta)}d\theta \\ =& \int \operatorname{sen}(\theta)\cos(\theta)d\theta + 3\int \cos^2(\theta)d\theta \\ =& \frac{\operatorname{sen}^2(\theta)}{2} + \frac{3}{2}(\theta + \operatorname{sen}(\theta)\cos(\theta)) + c \\ =& \frac{(x-2)^2}{2(x^2-4x+5)} + \frac{3}{2}\left(\arctan(x-2) + \frac{x-2}{x^2-4x+5}\right) + c\end{aligned}$$

-A- e B

Questão 2. a) (1,5 ponto) Seja $B = \{(x,y) \in \mathbb{R}^2 : 2x^2 + y^2 \leq 1 \text{ e } y \geq \sqrt{x}\}$. Calcule o volume do sólido obtido pela rotação de B em torno da reta $y = 2$.

b) (2,0 pontos) Calcule a área da região A formada pelos pontos $(x,y) \in \mathbb{R}^2$ tais que y está entre $f(x) = \sqrt{1-2x^2}$ e $g(x) = \sqrt{x}$, para $0 \leq x \leq \frac{1}{\sqrt{2}}$.

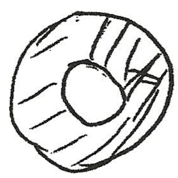
a)



$$2x^2 + y^2 = 1 \overset{y>0}{\Rightarrow} y = \sqrt{1-2x^2}$$

$$\begin{cases} 2x^2+y^2 = 1 \\ y = \sqrt{x} \Rightarrow y^2 = x \end{cases} \Rightarrow \begin{array}{l} 2x^2 + x = 1 \\ 2x^2 + x - 1 = 0 \end{array} \quad \boxed{x = 1/2} \quad \cancel{x = -1}$$

Calculando $A(x)$:

$2 - \sqrt{1-2x^2}$  $2 - \sqrt{x}$

$$A(x) = \pi(2-\sqrt{x})^2 - \pi(2-\sqrt{1-2x^2})^2 = \pi\left[4\sqrt{1-2x^2} + x - 4\sqrt{x} - 1 + 2x^2\right]$$

Logo

$$V = \int_0^{1/2} A(x)\,dx = 4\pi\int_0^{1/2} \sqrt{1-2x^2}\,dx + \pi\int_0^{1/2}(x - 4\sqrt{x} - 1 + 2x^2)\,dx$$

1) $$\int_0^{1/2} \sqrt{1-2x^2}\,dx = \int_0^{\pi/4} \sqrt{1-\operatorname{sen}^2 u}\,\frac{\cos u}{\sqrt{2}}\,du = \frac{1}{\sqrt{2}}\int_0^{\pi/4} \cos^2 u\,du = \frac{1}{\sqrt{2}}\int_0^{\pi/4}\left(\frac{1}{2} + \frac{\cos 2u}{2}\right)du =$$

$x = \frac{\operatorname{sen} u}{\sqrt{2}}$ $\quad -\pi/2 \leq u \leq \pi/2 \quad \therefore \cos u > 0 \Rightarrow \sqrt{1-\operatorname{sen}^2 u} = \cos u$

$dx = \frac{\cos u}{\sqrt{2}}$ $\quad x = 0 \Rightarrow u = 0 \quad x = \frac{1}{2} \Rightarrow \operatorname{sen} u = \frac{\sqrt{2}}{2} \Rightarrow u = \pi/4$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{2}u + \frac{\operatorname{sen} 2u}{4}\right)\Big|_0^{\pi/4} = \frac{1}{\sqrt{2}}\left(\frac{\pi}{8} + \frac{1}{4}\right)$$

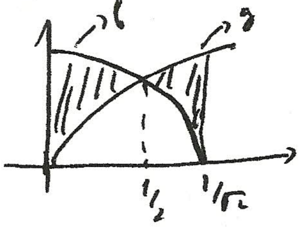
2) $$\int_0^{1/2} [x - 4\sqrt{x} - 1 + 2x^2]\,dx = \frac{x^2}{2} - 4\,\frac{2}{3}x^{3/2} - x + \frac{2x^3}{3}\Big|_0^{1/2} =$$

$$= \frac{1}{8} - \frac{8}{3}\cdot\frac{1}{2\sqrt{2}} - \frac{1}{2} + \frac{2}{3}\cdot\frac{1}{8} = -\frac{7}{24} - \frac{4}{3\sqrt{2}}$$

Logo

$$V = \frac{4\pi}{\sqrt{2}}\left(\frac{\pi}{8} + \frac{1}{4}\right) + \pi\left[-\frac{7}{24} - \frac{4}{3\sqrt{2}}\right]$$

b)



$f(x) = \sqrt{1-2x^2}$

$g(x) = \sqrt{x}$

Pelo item (a), $f(x) = g(x) \overset{x>0}{\Rightarrow} x = \frac{1}{2}$

Logo

$$A = \int_0^{1/2} \sqrt{1-2x^2}\,dx - \int_0^{1/2} \sqrt{x}\,dx + \int_{1/2}^{\frac{1}{\sqrt{2}}} \sqrt{x}\,dx - \int_{1/2}^{1/\sqrt{2}} \sqrt{1-2x^2}\,dx$$

Calculando as integrais:

I) $\displaystyle\int_0^{1/2} \sqrt{1-2x^2}\,dx = \frac{1}{\sqrt{2}}\left(\frac{\pi}{8} + \frac{1}{4}\right)$ (item (a))

II) $\displaystyle\int_0^{1/2} \sqrt{x}\,dx = \frac{2}{3} x^{3/2}\Big|_0^{1/2} = \frac{2}{3}\cdot\frac{1}{2\sqrt{2}} = \frac{1}{3\sqrt{2}}$

III) $\displaystyle\int_{1/2}^{1/\sqrt{2}} \sqrt{x}\,dx = \frac{2}{3} x^{3/2}\Big|_{1/2}^{1/\sqrt{2}} = \frac{2}{3}\left[\frac{1}{\sqrt[4]{8}} - \frac{1}{2\sqrt{2}}\right]$

IV) $\displaystyle\int_{1/2}^{1/\sqrt{2}} \sqrt{1-2x^2}\,dx = \int_{\pi/4}^{\pi/2} \sqrt{1-\text{sen}^2 u}\,\frac{\cos u}{\sqrt{2}}\,du = \frac{1}{\sqrt{2}}\int_{\pi/4}^{\pi/2} \cos^2 u\,du =$

$x = \frac{\text{sen}\,u}{\sqrt{2}}$ $\quad -\pi/2 \le u \le \pi/2 \quad \therefore \cos u > 0$

$x = 1/2 \Rightarrow \text{sen}\,u = \frac{\sqrt{2}}{2} \Rightarrow u = \pi/4$

$dx = \frac{\cos u}{\sqrt{2}}$

$x = \frac{1}{\sqrt{2}} \Rightarrow \text{sen}\,u = 1 \Rightarrow u = \pi/2$

$$= \frac{1}{\sqrt{2}}\left[\frac{1}{2}u + \frac{\text{sen}\,2u}{4}\right]\Big|_{\pi/4}^{\pi/2} = \frac{1}{\sqrt{2}}\left[\frac{1}{2}\frac{\pi}{2} + 0 - \frac{1}{2}\frac{\pi}{4} - \frac{1}{4}\right] = \frac{1}{\sqrt{2}}\left[\frac{\pi}{4} - \frac{1}{4}\right]$$

Logo

$$A = \frac{1}{\sqrt{2}}\left(\frac{\pi}{8} + \frac{1}{4}\right) - \frac{1}{3\sqrt{2}} + \frac{2}{3}\left[\frac{1}{\sqrt[4]{8}} - \frac{1}{2\sqrt{2}}\right] - \frac{1}{\sqrt{2}}\left[\frac{\pi}{4} - \frac{1}{4}\right].$$

Questão 3. (3,0 pontos) Considere $f : [0, +\infty[\longrightarrow \mathbb{R}$ dada por $f(x) = \int_0^x \sqrt{4e^{2t} - 1}\, dt$ e $g : \mathbb{R} \longrightarrow \mathbb{R}$ dada por $g(x) = \int_0^{\operatorname{sen} x} x^3 e^{t^2}\, dt$.

a) Calcule $g'(x)$.

b) Seja $F(x) = f(g(x))$. Calcule $F'(\pi)$.

c) Calcule o comprimento do gráfico de f entre $x = 0$ e $x = 1$.

a) $g(x) = x^3 \int_0^{\operatorname{sen} x} e^{t^2} dt$.

A função $x \mapsto e^{x^2}$ é contínua em $\mathbb{R}$. logo, as funções envolvidas na g são deriváveis. Usando as regras do produto e da cadeia e o TFC, obtemos

$$\boxed{g'(x) = 3x^2 \int_0^{\operatorname{sen} x} e^{t^2} dt + x^3 e^{\operatorname{sen}^2 x} \cos x.}$$ p/ todo $x \in \mathbb{R}$

b) A função $x \mapsto \sqrt{4e^{2x} - 1}$ é contínua em $[\ln \frac{1}{2}, +\infty[$.

Pelo TFC, $f'(x) = \sqrt{4e^{2x} - 1}$, $\forall x > \ln \frac{1}{2}$, em particular, para $x \geq 0$.

$g(\pi) = \pi^3 \int_0^0 e^{t^2} dt = 0$ e $g'(\pi) = 3\pi^2 \int_0^0 e^{t^2} dt + \pi^3 e^0 (-1) = -\pi^2$

f é derivável em $0 = g(\pi)$ e g é derivável em π.

Pela RC, $F(x) = f(g(x))$ é derivável em π e $F'(\pi) = f'(g(\pi)) g'(\pi)$, isto é,

$$F'(\pi) = f'(0)(-\pi^2) = \boxed{-\sqrt{3}\,\pi^2}$$

c) $f'(x) = \sqrt{9e^{2x} - 1}$ é contínua em $[0, 1]$

logo,

$$L = \int_0^1 \sqrt{1 + (f'(x))^2}\, dx = \int_0^1 \sqrt{4e^{2x}}\, dx = 2 \int_0^1 e^x dx = \boxed{2(e-1)}$$

Questão 1 (Turma B)

Item a) Aplicamos duas vezes a integração por partes:

$$\begin{aligned}\int x^3(\ln(x))^2dx =& \frac{x^4}{4}(\ln(x))^2 - \frac{1}{2}\int x^3\ln(x)dx\\ =& \frac{x^4}{4}(\ln(x))^2 - \frac{1}{2}(\frac{x^4}{4}\ln(x) - \frac{1}{4}\int x^3dx)\\ =& \frac{x^4}{4}(\ln(x))^2 - \frac{1}{8}x^4\ln(x) + \frac{1}{32}x^4 + c\end{aligned}$$

Item b) Usamos a substituição $u = x^4$ e, em seguida, $v = \tan(u)$:

$$\begin{aligned}\int x^3\tan^3(x^4)\sec^4(x^4)dx =& \frac{1}{4}\int \tan^3(u)\sec^4(u)du\\ =& \frac{1}{4}\int \tan^3(u)(1+\tan^2(u))\sec^2(u)du\\ =& \frac{1}{4}\int v^3(1+v^2)dv\\ =& \frac{1}{4}\left(\frac{v^4}{4}+\frac{v^6}{6}\right)+c\\ =& \frac{\tan^4(x^4)}{16}+\frac{\tan^6(x^4)}{24}+c\end{aligned}$$

Item c) Usamos a substituição trigonométrica $x + 2 = \tan(\theta)$:

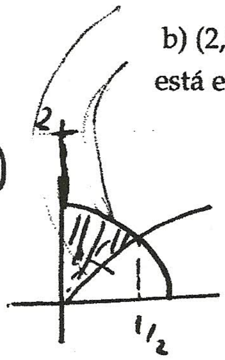
$$\begin{aligned}\int \frac{x+3}{(x^2+4x+5)^2}dx &= \int \frac{x+3}{((x+2)^2+1)^2}dx\\ &= \int \frac{\tan(\theta)+1}{(\tan^2(\theta)+1)^2}\sec^2(\theta)d\theta\\ &= \int \frac{\tan(\theta)+1}{\sec^2(\theta)}d\theta\\ &= \int \operatorname{sen}(\theta)\cos(\theta)d\theta + \int \cos^2(\theta)d\theta\\ &= \frac{\operatorname{sen}^2(\theta)}{2} + \frac{1}{2}(\theta + \operatorname{sen}(\theta)\cos(\theta)) + c\\ &= \frac{(x+2)^2}{2(x^2+4x+5)} + \frac{1}{2}\left(\arctan(x+2) + \frac{x+2}{x^2+4x+5}\right) + c\end{aligned}$$

-A- e B

Questão 2. a) (1,5 ponto) Seja $B = \{(x,y) \in \mathbb{R}^2 : 2x^2 + y^2 \leq 1 \text{ e } y \geq \sqrt{x}\}$. Calcule o volume do sólido obtido pela rotação de B em torno da reta $y = 2$.

b) (2,0 pontos) Calcule a área da região A formada pelos pontos $(x,y) \in \mathbb{R}^2$ tais que y está entre $f(x) = \sqrt{1-2x^2}$ e $g(x) = \sqrt{x}$, para $0 \leq x \leq \frac{1}{\sqrt{2}}$.

a)



$2x^2 + y^2 = 1 \overset{y>0}{\Rightarrow} y = \sqrt{1-2x^2}$

$\begin{cases} 2x^2+y^2=1 \\ y=\sqrt{x} \Rightarrow y^2 = x \end{cases} \Rightarrow \begin{array}{l} 2x^2 + x = 1 \\ 2x^2 + x - 1 = 0 \end{array} \quad \boxed{x = 1/2} \quad \cancel{x = -1}$

Calculando $A(x)$:

$A(x) = \pi(2-\sqrt{x})^2 - \pi(2-\sqrt{1-2x^2})^2 = \pi\left[4\sqrt{1-2x^2} + x - 4\sqrt{x} - 1 + 2x^2\right]$

Logo

$$V = \int_0^{1/2} A(x)\,dx = 4\pi\int_0^{1/2}\sqrt{1-2x^2}\,dx + \pi\int_0^{1/2}(x - 4\sqrt{x} - 1 + 2x^2)\,dx$$

1) $\displaystyle\int_0^{1/2}\sqrt{1-2x^2}\,dx = \int_0^{\pi/4}\sqrt{1-\text{sen}^2 u}\,\frac{\cos u}{\sqrt{2}}\,du = \frac{1}{\sqrt{2}}\int_0^{\pi/4}\cos^2 u\,du = \frac{1}{\sqrt{2}}\int_0^{\pi/4}\left(\frac{1}{2} + \frac{\cos 2u}{2}\right)du =$

$x = \frac{\text{sen}\, u}{\sqrt{2}}$ $\quad -\pi/2 \leq u \leq \pi/2 \quad \therefore \cos u > 0 \Rightarrow \sqrt{1-\text{sen}^2 u} = \cos u$

$dx = \frac{\cos u}{\sqrt{2}}$ $\quad x = 0 \Rightarrow u = 0 \quad x = \frac{1}{2} \Rightarrow \text{sen}\, u = \frac{\sqrt{2}}{2} \Rightarrow u = \frac{\pi}{4}$

$= \frac{1}{\sqrt{2}}\left(\frac{1}{2}u + \frac{\text{sen}\, 2u}{4}\right)\Big|_0^{\pi/4} = \frac{1}{\sqrt{2}}\left(\frac{\pi}{8} + \frac{1}{4}\right)$

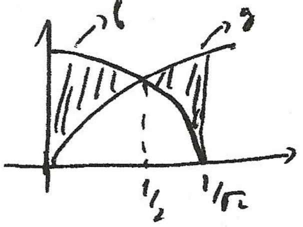
2) $\displaystyle\int_0^{1/2}[x - 4\sqrt{x} - 1 + 2x^2]\,dx = \frac{x^2}{2} - 4\,\frac{2}{3}x^{3/2} - x + \frac{2x^3}{3}\Big|_0^{1/2} =$

$= \frac{1}{8} - \frac{8}{3}\cdot\frac{1}{2\sqrt{2}} - \frac{1}{2} + \frac{2}{3}\cdot\frac{1}{8} = -\frac{7}{24} - \frac{4}{3\sqrt{2}}$

Logo

$$V = \frac{4\pi}{\sqrt{2}}\left(\frac{\pi}{8} + \frac{1}{4}\right) + \pi\left[-\frac{7}{24} - \frac{4}{3\sqrt{2}}\right]$$

b)



$f(x) = \sqrt{1-2x^2}$

$g(x) = \sqrt{x}$

Pelo item (a), $f(x) = g(x) \overset{x>0}{\Rightarrow} x = \frac{1}{2}$

Logo

$$A = \int_0^{1/2} \sqrt{1-2x^2}\,dx - \int_0^{1/2} \sqrt{x}\,dx + \int_{1/2}^{\frac{1}{\sqrt{2}}} \sqrt{x}\,dx - \int_{1/2}^{1/\sqrt{2}} \sqrt{1-2x^2}\,dx$$

Calculando as integrais:

I) $\displaystyle\int_0^{1/2} \sqrt{1-2x^2}\,dx = \frac{1}{\sqrt{2}}\left(\frac{\pi}{8} + \frac{1}{4}\right)$ (item (a))

II) $\displaystyle\int_0^{1/2} \sqrt{x}\,dx = \frac{2}{3} x^{3/2} \Big|_0^{1/2} = \frac{2}{3} \cdot \frac{1}{2\sqrt{2}} = \frac{1}{3\sqrt{2}}$

III) $\displaystyle\int_{1/2}^{1/\sqrt{2}} \sqrt{x}\,dx = \frac{2}{3} x^{3/2} \Big|_{1/2}^{1/\sqrt{2}} = \frac{2}{3}\left[\frac{1}{\sqrt[4]{8}} - \frac{1}{2\sqrt{2}}\right]$

IV) $\displaystyle\int_{1/2}^{1/\sqrt{2}} \sqrt{1-2x^2}\,dx = \int_{\pi/4}^{\pi/2} \sqrt{1-\operatorname{sen}^2 u}\,\frac{\cos u}{\sqrt{2}}\,du = \frac{1}{\sqrt{2}} \int_{\pi/4}^{\pi/2} \cos^2 u\,du =$

$x = \dfrac{\operatorname{sen} u}{\sqrt{2}}$ $\quad -\pi/2 \leq u \leq \pi/2 \quad \therefore \cos u > 0$

$dx = \dfrac{\cos u}{\sqrt{2}}$

$x = 1/2 \Rightarrow \operatorname{sen} u = \frac{\sqrt{2}}{2} \Rightarrow u = \pi/4$

$x = \frac{1}{\sqrt{2}} \Rightarrow \operatorname{sen} u = 1 \Rightarrow u = \pi/2$

$$= \frac{1}{\sqrt{2}}\left[\frac{1}{2}u + \frac{\operatorname{sen} 2u}{4}\right]\Bigg|_{\pi/4}^{\pi/2} = \frac{1}{\sqrt{2}}\left[\frac{1}{2}\frac{\pi}{2} + 0 - \frac{1}{2}\frac{\pi}{4} - \frac{1}{4}\right] = \frac{1}{\sqrt{2}}\left[\frac{\pi}{4} - \frac{1}{4}\right]$$

Logo

$$A = \frac{1}{\sqrt{2}}\left(\frac{\pi}{8} + \frac{1}{4}\right) - \frac{1}{3\sqrt{2}} + \frac{2}{3}\left[\frac{1}{\sqrt[4]{8}} - \frac{1}{2\sqrt{2}}\right] - \frac{1}{\sqrt{2}}\left[\frac{\pi}{4} - \frac{1}{4}\right].$$

Questão 3. (3,0 pontos) Considere $f : [0, +\infty[\longrightarrow \mathbb{R}$ dada por $f(x) = \int_0^x \sqrt{9e^{2t} - 1}\, dt$ e $g : \mathbb{R} \longrightarrow \mathbb{R}$ dada por $g(x) = \int_0^{\operatorname{sen} x} x^2 e^{t^2}\, dt$.

a) Calcule $g'(x)$.

b) Seja $F(x) = f(g(x))$. Calcule $F'(\pi)$.

c) Calcule o comprimento do gráfico de f entre $x = 0$ e $x = 1$.

a) $g(x) = x^2 \int_0^{\operatorname{sen} x} e^{t^2} dt.$

Observe que a função $x \mapsto e^{x^2}$ é contínua em $\mathbb{R}$. Logo, as funções envolvidas são deriváveis.

Usando as regras do produto e da cadeia e o TFC, obtemos, para $x \in \mathbb{R}$:

$$\boxed{g'(x) = 2x \int_0^{\operatorname{sen} x} e^{t^2} dt + x^2 e^{\operatorname{sen}^2 x} \cos x}$$

b) A função $x \mapsto \sqrt{9e^{2x} - 1}$ é contínua em $[\ln \frac{1}{3}, +\infty[$

Pelo TFC, $f'(x) = \sqrt{9e^{2x} - 1}$, $\forall x > \ln \frac{1}{3}$, em particular, p/ $x \geq 0$.

$g(\pi) = \pi^2 \int_0^0 e^{t^2} dt = 0$ e $g'(\pi) = 2\pi \int_0^0 e^{t^2} dt + \pi^2 e^0 (-1) = -\pi^2$

f é derivável em $0 = g(\pi)$ e g é derivável em π.

Pela RC, $F(x) = f(g(x))$ é derivável em π e $F'(\pi) = f'(g(\pi))g'(\pi)$, isto é,

$$F'(\pi) = f'(0)(-\pi^2) = \boxed{-\sqrt{8}\pi^2}$$

c) $f'(x) = \sqrt{9e^{2x} - 1}$ é contínua em $[0, 1]$.

Logo,

$$L = \int_0^1 \sqrt{1 + (f'(x))^2}\, dx = \int_0^1 \sqrt{9e^{2x}}\, dx = 3 \int_0^1 e^x dx = \boxed{3(e - 1)}$$