Gabarito da Terceira Prova
MAT-2453 - Tipos A e B

22 de Junho de 2011

Questão 1. Calcule as seguintes integrais indefinidas:

a) (1,0 ponto) $\int x \operatorname{arcsen}(5x^2)dx$

b) (1,5 ponto) $\int \frac{1}{x^4\sqrt{x^2+9}}dx$

a) $\int x \operatorname{arcsen}(5x^2)\,dx = \frac{x^2}{2}\operatorname{arcsen}(5x^2) - \int \frac{5x^3}{\sqrt{1-25x^4}}\,dx$

$$\left(\begin{array}{l} u = \operatorname{arcsen}(5x^2) \rightarrow du = \frac{1}{\sqrt{1-25x^4}} \cdot 10x\,dx \\ dv = x\,dx \rightarrow v = \frac{x^2}{2} \end{array} \right)$$

$$\int \frac{5x^3}{\sqrt{1-25x^4}}\,dx = -\frac{1}{20}\int \frac{dt}{\sqrt{t}} = -\frac{1}{20}\cdot\frac{t^{1/2}}{1/2} + C = -\frac{1}{10}\sqrt{t} + C = -\frac{1}{10}\sqrt{1-25x^4} + C.$$

$t = 1-25x^4 \Rightarrow dt = -4.25x^3dx$

$$\therefore \boxed{\int x \operatorname{arcsen}(5x^2)\,dx = \frac{x^2}{2}\operatorname{arcsen}(5x^2) + \frac{1}{10}\sqrt{1-25x^4} + C}$$

b) $x = 3\operatorname{tg}\theta, \quad \theta \in \left]-\pi/2, \pi/2\right[$

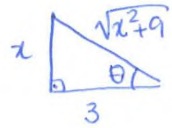
$dx = 3\sec^2\theta\, d\theta \qquad \sqrt{x^2+9} = 3|\sec\theta| = 3\sec\theta \qquad \theta \in \left]-\pi/2, \pi/2\right[$

$$\boxed{\int \frac{1}{x^4\sqrt{x^2+9}}\,dx} = \int \frac{3\sec^2\theta}{3^4\operatorname{tg}^4\theta \cdot 3\sec\theta}\,d\theta = \frac{1}{81}\int \frac{\cos^4\theta}{\operatorname{sen}^4\theta}\cdot\frac{1}{\cos\theta}\,d\theta = \frac{1}{81}\int \frac{\cos^3\theta}{\operatorname{sen}^4\theta}\,d\theta$$

$$= \frac{1}{81}\int \frac{1-\operatorname{sen}^2\theta}{\operatorname{sen}^4\theta}\cos\theta\,d\theta = \frac{1}{81}\int \frac{1-t^2}{t^4}\,dt = \frac{1}{81}\int (t^{-4} - t^{-2})\,dt = \frac{1}{81}\left(\frac{t^{-3}}{-3} - \frac{t^{-1}}{-1}\right) + C$$

$t = \operatorname{sen}\theta \Rightarrow dt = \cos\theta\, d\theta$

$$= \frac{1}{81}\left(-\frac{1}{3t^3} + \frac{1}{t}\right) + C = \frac{1}{81}\left(-\frac{1}{3\operatorname{sen}^3\theta} + \frac{1}{\operatorname{sen}\theta}\right) + C = \boxed{\frac{1}{81}\left(-\frac{\sqrt{x^2+9}^{\,3}}{3x^3} + \frac{\sqrt{x^2+9}}{x}\right) + C}$$



$\operatorname{sen}\theta = \frac{x}{\sqrt{x^2+9}}$

Questão 1. Calcule as seguintes integrais indefinidas:

a) (1,0 ponto) $\int x \operatorname{arcsen}(3x^2)dx$

b) (1,5 ponto) $\int \frac{1}{x^4\sqrt{x^2+4}}dx$

a) $\int x \operatorname{arcsen}(3x^2)dx = \frac{x^2}{2}\operatorname{arcsen}(3x^2) - \int \frac{3x^2}{\sqrt{1-9x^4}}dx$

$$\left(\begin{array}{ll} u = \operatorname{arcsen}(3x^2) & \rightarrow du = \frac{1}{\sqrt{1-9x^4}}.6x\,dx \\ dv = x\,dx & \rightarrow v = \frac{x^2}{2} \end{array} \right)$$

$$\int \frac{3x^3}{\sqrt{1-9x^4}}dx = -\frac{1}{12}\int \frac{dt}{\sqrt{t}} = -\frac{1}{12}\frac{t^{1/2}}{1/2} + C = -\frac{1}{6}\sqrt{t} + C = -\frac{1}{6}\sqrt{1-9x^4} + C$$

$t = 1-9x^4 \Rightarrow dt = -9.4.x^3dx$

$$\therefore \boxed{\int x \operatorname{arcsen}(3x^2)dx = \frac{x^2}{2}\operatorname{arcsen}(3x^2) + \frac{1}{6}\sqrt{1-9x^4} + C}$$

b) $x = 2\operatorname{tg}\theta,\ \theta \in \left]-\pi/2, \pi/2\right[$

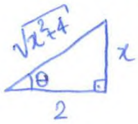
$dx = 2\sec^2\theta\,d\theta, \quad \sqrt{x^2+4} = 2|\sec\theta| = 2\sec\theta$ $\quad \theta \in \left]-\pi/2, \pi/2\right[$.

$$\boxed{\int \frac{1}{x^4\sqrt{x^2+4}}dx} = \int \frac{2\sec^2\theta}{2^4.\operatorname{tg}^4\theta.\,2\sec\theta}d\theta = \frac{1}{16}\int \frac{\cos^4\theta}{\operatorname{sen}^4\theta}.\frac{1}{\cos\theta}d\theta = \frac{1}{16}\int \frac{\cos^3\theta}{\operatorname{sen}^4\theta}d\theta$$

$$= \frac{1}{16}\int \frac{1-\operatorname{sen}^2\theta}{\operatorname{sen}^4\theta}\cos\theta\,d\theta = \frac{1}{16}\int \frac{1-t^2}{t^4}dt = \frac{1}{16}\int (t^{-4}-t^{-2})dt = \frac{1}{16}\left(\frac{t^{-3}}{-3} - \frac{t^{-1}}{-1}\right) + C$$

$t = \operatorname{sen}\theta \Rightarrow dt = \cos\theta\,d\theta$

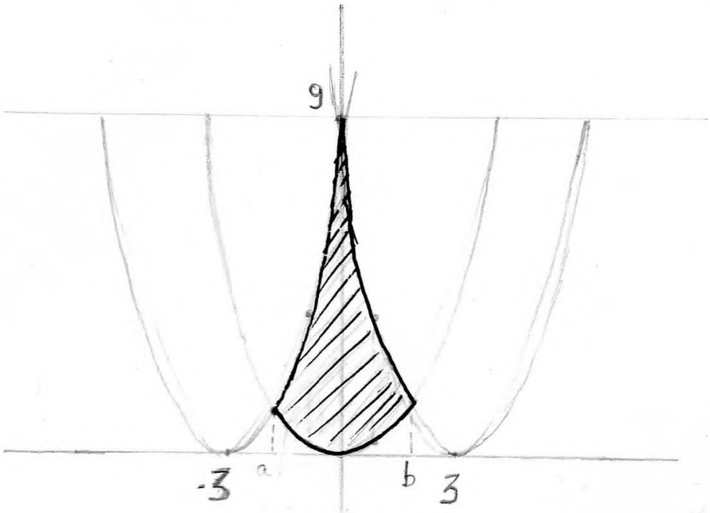
$$= \frac{1}{16}\left(\frac{1}{t} - \frac{1}{3t^3}\right) + C = \frac{1}{16}\left(\frac{1}{\operatorname{sen}\theta} - \frac{1}{3\operatorname{sen}^3\theta}\right) + C = \boxed{\frac{1}{16}\left(\frac{\sqrt{x^2+4}}{x} - \frac{\sqrt{(x^2+4)^3}}{3x^3}\right) + C}$$



$\operatorname{sen}\theta = \frac{x}{\sqrt{x^2+4}}$

Questão 2. (2,5 pontos) Determine o volume do sólido obtido pela rotação, em torno da reta $y = 9$, da região

$$A = \{(x, y) \in \mathbb{R}^2 : y \geq x^2, y \leq (x-3)^2, y \leq (x+3)^2\}$$



$a^2 = (a+3)^2 \Rightarrow$

$+6a + 9 = 0 \Rightarrow \boxed{a = -3/2}$

$b^2 = (b-3)^2 \Rightarrow$

$-6b + 9 = 0 \Rightarrow \boxed{b = 3/2}$

$$V = \int_a^0 \pi (9 - x^2)^2 dx - \int_a^0 \pi (9 - (x+3)^2)^2 dx + \int_0^b \pi (9 - x^2)^2 dx - \int_0^b \pi (9 - (x-3)^2)^2 dx$$

$$= 2\pi \int_0^b (9 - x^2)^2 - (9 - (x-3)^2)^2 dx \quad (*)$$

$$= 2\pi \int_0^{3/2} 81 - 54x^2 + 12x^3 dx = 2\pi \left[81x - 18x^3 + 3x^4 \right]_0^{3/2} = \frac{1215}{8}\pi$$

$(*)$ Fazendo $u = -x$ temos $du = -dx$ e

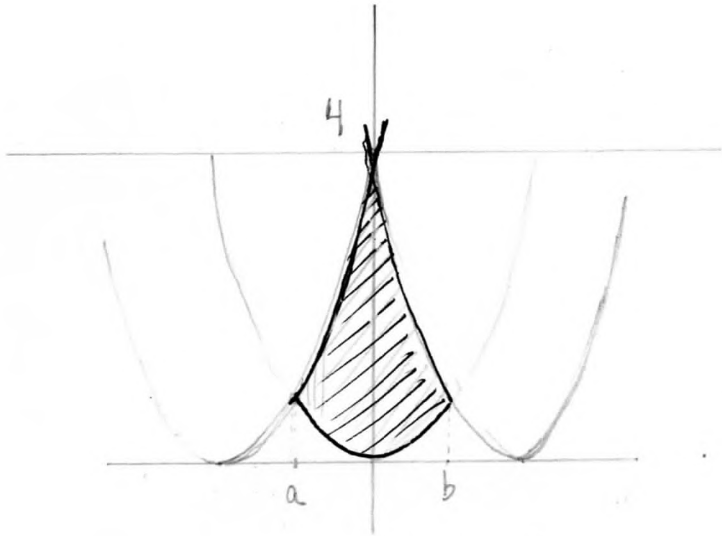
$$\pi \int_{-3/2}^{0} (9 - x^2)^2 - (9 - (x+3)^2)^2 dx = -\pi \int_{3/2}^{0} (9 - u^2)^2 - (9 - (u-3)^2)^2 du$$

a''

$$= \pi \int_0^{3/2} (9 - u^2)^2 - (9 - (u-3)^2)^2 du.$$

Questão 2. (2,5 pontos) Determine o volume do sólido obtido pela rotação, em torno da reta $y = 4$, da região

$$A = \{(x, y) \in \mathbb{R}^2 : y \geq x^2, y \leq (x-2)^2, y \leq (x+2)^2\}$$



$a^2 = (a+2)^2 \Rightarrow$

$+4a+4 = 0 \Rightarrow \boxed{a = -1}$

$b^2 = (b-2)^2 \Rightarrow$

$-4b+4 = 0 \Rightarrow \boxed{b = 1}$

$$V = \int_a^0 \pi (4-x^2)^2 dx - \int_a^0 \pi (4-(x+2)^2)^2 dx + \int_0^b \pi (4-x^2)^2 dx - \int_0^b \pi (4-(x-2)^2)^2 dx$$

$$= 2\pi \int_0^b (4-x^2)^2 - (4-(x-2)^2)^2 dx \quad (*)$$

$$= 2\pi \int_0^1 16 - 24x^2 + 8x^3 dx = 2\pi \left[16x - 8x^3 + 2x^4\right]_0^1 = 20\pi.$$

(*) Fazendo $u = -x$ temos $du = -dx$ e

$$\pi \int_{\underset{\overset{\shortparallel}{a}}{-1}}^0 (4-x^2)^2 - (4-(x+2)^2)^2 dx = -\pi \int_1^0 (4-u^2)^2 - (4-(u-2)^2)^2 du$$

$$= \pi \int_0^1 (4-u^2)^2 - (4-(u-2)^2)^2 du.$$

Questão 3.(turma A)

a) (1,5) **Calcule o comprimento de** $y(x) = \ln(1 - x^2)$ **para** $x \in [0, \frac{1}{2}]$**.**

b) (1,0) **Seja** $G(x) = \int_0^x (t \int_0^t e^{u^2} du)dt$**. Calcule** $G'(x)$ **e** $G''(x)$**.**

Solução:

$$
\begin{aligned}
a)\, \mathcal{L} &= \int_0^{\frac{1}{2}} \sqrt{1 + (y'(x))^2}\, dx = \int_0^{\frac{1}{2}} \sqrt{1 + (\frac{-2x}{1-x^2})^2}\, dx = \int_0^{\frac{1}{2}} \frac{1+x^2}{1-x^2}\, dx \\
&= \int_0^{\frac{1}{2}} \left(-1 + \frac{2}{1-x^2}\right) dx = -\int_0^{\frac{1}{2}} dx + 2\int_0^{\frac{1}{2}} \frac{1}{(1-x)(1+x)}\, dx \\
&= (-x)\Big|_0^{1/2} + \int_0^{\frac{1}{2}} \frac{1}{1-x} dx + \int_0^{\frac{1}{2}} \frac{1}{1+x} dx \\
&= -\frac{1}{2} \;-\; \ln(|1-x|)\Big|_0^{1/2} \;+\; \ln(|1+x|)\Big|_0^{1/2} = -\frac{1}{2} + \ln(3)
\end{aligned}
$$

$$
b)\, G'(x) = x\int_0^x e^{u^2} du \quad \& \quad G''(x) = \int_0^x e^{u^2} du + xe^{x^2}
$$

Questão 3.(turma B)

a) (1,5) **Calcule o comprimento de** $y(x) = \ln(1 - x^2)$ **para** $x \in [0, \frac{1}{3}]$**.**

b) (1,0) **Seja** $G(x) = \int_0^x (t \int_0^t \sin(u^2) du)dt$**. Calcule** $G'(x)$ **e** $G''(x)$**.**

Solução:

$$
\begin{aligned}
a)\, \mathcal{L} &= \int_0^{\frac{1}{3}} \sqrt{1 + (y'(x))^2}\, dx = \int_0^{\frac{1}{3}} \sqrt{1 + (\frac{-2x}{1-x^2})^2}\, dx = \int_0^{\frac{1}{3}} \frac{1+x^2}{1-x^2}\, dx \\
&= \int_0^{\frac{1}{3}} \left(-1 + \frac{2}{1-x^2}\right) dx = -\int_0^{\frac{1}{3}} dx + 2\int_0^{\frac{1}{3}} \frac{1}{(1-x)(1+x)}\, dx \\
&= (-x)\Big|_0^{1/3} + \int_0^{\frac{1}{3}} \frac{1}{1-x} dx + \int_0^{\frac{1}{3}} \frac{1}{1+x} dx \\
&= -\frac{1}{3} \;-\; \ln(|1-x|)\Big|_0^{1/3} \;+\; \ln(|1+x|)\Big|_0^{1/3} = -\frac{1}{3} + \ln(2)
\end{aligned}
$$

$$
b)\, G'(x) = x\int_0^x \sin(u^2) du \quad \& \quad G''(x) = \int_0^x \sin(u^2) du + x\sin(x^2)
$$

Questão 4. Seja $f(x) = \ln(3x+1)$, com $x > -\frac{1}{3}$.

a) (1,0 ponto) Determine o polinômio de Taylor de ordem 3 de f em torno do ponto $x_0 = 0$.

b) (1,5 ponto) Usando o item (a), calcule um valor aproximado para $\ln(1,03)$ e para $\ln(0,97)$. Verifique que o erro em ambas as aproximações é menor do que 10^{-6}.

(a) $f(x) = \ln(3x+1)$ $\quad f(0) = \ln 1 = 0$

$f'(x) = \dfrac{3}{3x+1}$ $\quad f'(0) = 3$

$f''(x) = \dfrac{-9}{(3x+1)^2}$ $\quad f''(0) = -9$

$f'''(x) = \dfrac{54}{(3x+1)^3}$ $\quad f'''(0) = 54$

Logo

$$P(x) = 3x - \frac{9x^2}{2} + \frac{54x^3}{3!}$$

$$\boxed{P(x) = 3x - \frac{9x^2}{2} + 9x^3}$$

(b) . $\ln(1,03)$: considere $x = 0,01$. Daí $\ln(1,03) \approx P(0,01) = 3.10^{-2} - \frac{9}{2}10^{-4} + 9.10^{-6}$

$\therefore \ln(1,03) \approx 0,029559$

. $\ln(0,97)$ considere $x = -0,01$. Daí $\ln(0,97) \approx P(-0,01) = -3.10^{-2} - \frac{9}{2}.10^{-4} - 9.10^{-6}$

$\therefore \ln(0,97) \approx -0,03054$

Erro $\quad f^{(4)}(x) = \dfrac{-54.9}{(3x+1)^4}$ $\quad \therefore \quad |E(x)| = \dfrac{54.9}{(3c+1)^4} \cdot \dfrac{1}{4!} x^4 = \dfrac{\overset{81}{27.3}}{4} \cdot \dfrac{1}{(3c+1)^4} x^4$, para algum c entre 0 e x

. Se $x = 0,01$, teremos $c > 0$ e então $3c+1 > 1$. Logo $\dfrac{1}{(3c+1)^4} < 1$

$$|E(0,01)| \leq \frac{\overset{81}{27.3}}{4} . 10^{-8} < 10^{-6}$$

. Se $x = -0,01$, teremos que $c > -0,01$ e portanto $3c+1 > 0,97 > 0,9$

$(0,9)^2 = 0,81 > 0,8$

Note que $(3c+1)^4 > (0,9)^4 > (0,8)^2 = 0,64 > 0,5$.

Logo $\dfrac{1}{(3c+1)^4} < 2$

Teremos então que $|E(-0,01)| \leq \dfrac{81}{\cancel{4}_2} . \cancel{2} . 10^{-8} < 10^{-6}$

Questão 4. Seja $f(x) = \ln(2x+1)$, com $x > -\frac{1}{2}$.

a) (1,0 ponto) Determine o polinômio de Taylor de ordem 3 de f em torno do ponto $x_0 = 0$.

b) (1,5 ponto) Usando o item (a), calcule um valor aproximado para $\ln(1,02)$ e para $\ln(0,98)$. Verifique que o erro em ambas as aproximações é menor do que 10^{-7}.

(a) $f(x) = \ln(2x+1)$ $\quad f(0) = \ln 1 = 0$

$f'(x) = \frac{2}{2x+1}$ $\quad f'(0) = 2$

$f''(x) = \frac{-4}{(2x+1)^2}$ $\quad f''(0) = -4$

$f'''(x) = \frac{16}{(2x+1)^3}$ $\quad f'''(0) = 16$

Logo

$P(x) = 2x - \frac{4x^2}{2} + \frac{16}{3!}x^3$

$$\boxed{P(x) = 2x - 2x^2 + \frac{8}{3}x^3}$$

(b) . $\ln(1,02)$: considere $x = 0,01$. Daí $\ln(1,02) \approx P(0,01) = 2.10^{-2} - 2.10^{-4} + \frac{8}{3}$

$\therefore \ln(1,02) \approx 0,0198026666$

. $\ln(0,98)$: considere $x = -0,01$. Daí $\ln(0,98) \approx P(-0,01) = -2.10^{-2} - 2.10^{-4} - \frac{8}{3}$

$\therefore \ln(0,98) \approx -0,0202026666$

Erro $f^{(4)}(x) = -\frac{16.3.2}{(2x+1)^4}$ $\quad |E(x)| = \frac{\cancel{16.3.2}\ 4}{(2c+1)^4} \cdot \frac{1}{\cancel{4!}} x^4 = \frac{4}{(2c+1)^4} \cdot x^4$, para algum c entre 0 e x

. Se $x = 0,01$, teremos $c > 0$ e então $2c+1 > 1$. Logo $\frac{1}{(2c+1)^4} < 1$

$\therefore |E(0,01)| \leq 4.10^{-8} < 10^{-7}$

. Se $x = -0,01$, teremos que $c > -0,01$ e portanto $2c+1 > 0,98 > 0,9$

Note que $(2c+1)^4 > (0,9)^4 > (0,8)^2 = 0,64 > 0,5$ $\quad (0,9)^2 = 0,81 > 0,8$

Logo $\frac{1}{(2c+1)^4} < 2$

Teremos então que $|E(0,01)| \leq 4.2.(-10^{-2})^4 = 8.10^{-8} < 10^{-7}$