Gabarito da Questão 1 - Prova Tipo A / 2010

Calcule as seguintes integrais indefinidas:

(1,5) *a*) $\int \frac{x^5}{\sqrt{3+x^2}}dx$

Fazendo a mudança de variáveis $x = \sqrt{3}\tan u$, temos que $dx = \sqrt{3}\sec^2 u du$, logo

$$\begin{aligned}
\int \frac{x^5}{\sqrt{3+x^2}}dx &= 9\sqrt{3}\int \tan^5 u \sec u du \\
&= 9\sqrt{3}\int (\sec^2 u - 1)^2(\sec u \tan u)du \\
&= 9\sqrt{3}\int (\sec^4 u - 2\sec^2 u + 1)(\sec u \tan u)du \\
&= 9\sqrt{3}\left(\frac{\sec^5 u}{5} - 2\frac{\sec^3 u}{3} + \sec u\right) + C \\
&= 9\sqrt{3}\left\{\frac{1}{5}\left(1+\frac{x^2}{3}\right)^{5/2} - \frac{2}{3}\left(1+\frac{x^2}{3}\right)^{3/2} + \left(1+\frac{x^2}{3}\right)^{1/2}\right\} + C \\
&= \frac{1}{5}(3+x^2)^{5/2} - 2(3+x^2)^{3/2} + 9(3+x^2)^{1/2} + C\,.
\end{aligned}$$

(2,0) *b*) $\int \frac{3e^x}{(e^x+2)(e^{2x}+2e^x+3)}dx$

Fazendo a mudança de variáveis $e^x = u$, temos que $e^x dx = du$, logo

$$\int \frac{3e^x}{(e^x+2)(e^{2x}+2e^x+3)}dx = \int \frac{3}{(u+2)(u^2+2u+3)}dx\,. \tag{1}$$

Vamos agora encontrar $A, B, C \in \mathbb{R}$ tais que

$$\frac{3}{(u+2)(u^2+2u+3)} = \frac{A}{u+2} + \frac{Bu+C}{u^2+2u+3}\,.$$

Isso implica que A, B, C satisfazem o sistema linear

$$\begin{cases} A + B = 0 \\ 2A + 2B + C = 0 \\ 3A + 2C = 3 \end{cases},$$

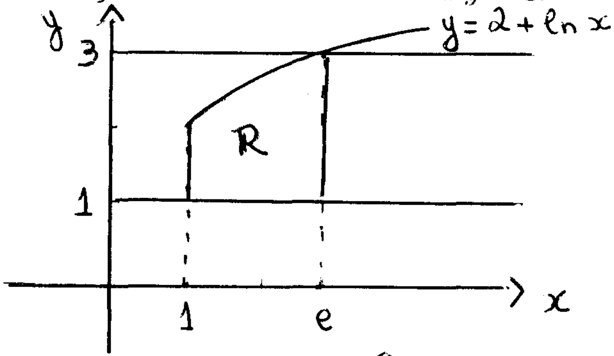
e portanto, $A = 1$, $B = -1$ e $C = 0$. Voltando à equação (1), temos que

$$\begin{aligned}
\int \frac{3e^x}{(e^x+2)(e^{2x}+2e^x+3)}dx &= \int \left(\frac{1}{u+2} - \frac{u}{u^2+2u+3}\right)du \\
&= \ln|u+2| - \frac{1}{2}\int \frac{u}{\left(\frac{u+1}{\sqrt{2}}\right)^2 + 1}du \\
&= \ln|u+2| - \frac{1}{\sqrt{2}}\int \frac{\sqrt{2}v - 1}{v^2+1}dv\,, \quad \text{onde } v = \frac{u+1}{\sqrt{2}} \\
&= \ln|u+2| - \frac{1}{2}\ln(1+v^2) + \frac{1}{\sqrt{2}}\arctan v + C \\
&= \ln(e^x+2) - \frac{1}{2}\ln\left(1 + \left(\frac{e^x+1}{\sqrt{2}}\right)^2\right) + \frac{1}{\sqrt{2}}\arctan\left(\frac{e^x+1}{\sqrt{2}}\right) + C
\end{aligned}$$

A

Questão 2. (2,5)

Seja $R = \{(x,y) \in \mathbb{R}^2 : 1 \le x \le e,\ 1 \le y \le 2 + \ln(x)\}$. Calcule o volume do sólido obtido pela rotação de R em torno da reta $y = 3$.



$$V = \pi \int_1^e \left[(3-1)^2 - (3-2-\ln x)^2\right] dx = \pi \int_1^e (3 + 2\ln x - \ln^2 x)\, dx =$$

$$= \pi \left[3(e-1) + 2\int_1^e \ln x \, dx - \int_1^e \ln^2 x \, dx \right]$$

Cálculo das integrais (por partes)

$$\int_1^e 1 . \ln x \, dx = x \ln x \Big|_1^e - \int_1^e x . \frac{1}{x} \, dx = (x \ln x - x) \Big|_1^e =$$

$$= (e \ln e - e) - (1 \ln 1 - 1) = 1$$

$$\int_1^e 1 \ln^2 x \, dx = x \ln^2 x \Big|_1^e - \int_1^e x (2 \ln x) \frac{1}{x} \, dx = x \ln^2 x \Big|_1^e - 2 \int_1^e \ln x \, dx =$$

$$= (e \ln^2 e - 1 \ln^2 1) - 2 = e - 2$$

Resposta: $V = \pi \left[3(e-1) + 2 - (e-2) \right] = \pi (2e + 1)$

3. Considere a função $g(x) = \int_0^{2\cos(x)} [3 + \text{sen}(t^2)]dt$ definida em $\mathbb{R}$.

(1,0) a) Calcule $g'(x)$.

(1,0) b) Seja $f(x) = \int_0^{g(x)} \frac{x^2}{\sqrt{4+t^4}}dt$. Calcule $f'(\pi/2)$.

Solução:

a) Usando o segundo teorema fundamental do Cálculo, obtemos

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left(\int_0^{2\cos(x)} [3 + \text{sen}(t^2)]dt\right) \\ &= \left[3 + \text{sen}(4\cos^2(x))\right](-2\text{sen}(x)) \end{aligned}$$

b) Novamente, pelo segundo teorema fundamental do Cálculo

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\int_0^{g(x)} \frac{x^2}{\sqrt{4+t^4}}dt\right) \\ &= \frac{d}{dx}\left(x^2\int_0^{g(x)} \frac{1}{\sqrt{4+t^4}}dt\right) \\ &= 2x\int_0^{g(x)} \frac{1}{\sqrt{4+t^4}}dt + x^2\left(\frac{1}{\sqrt{4+(g(x))^4}}\right)g'(x) \end{aligned}$$

Fazendo $x = \pi/2$ na expressão acima e observando que $g(\pi/2) = 0$, obtemos

$$f'(\pi/2) = \frac{\pi^2}{4} \cdot \frac{1}{2} \cdot (-6) = -\frac{3\pi^2}{4}$$

Questão 4.

(1,0) a) Mostre que para todo $x \in \mathbb{R}$ temos

$$\left|\text{sen}(x^4) - (x^4 - \frac{x^{12}}{3!} + \frac{x^{20}}{5!})\right| \leq \frac{|x|^{24}}{6!}$$

Sejam $f(y) = \text{sen}(y)$ e $y_0 = 0$.
O polinômio de Taylor de f de grau 5, em torno de y_0 é dado por:

$$p_5(y) = f(0)+f'(0)(y-0)+\frac{f''(0)}{2!}(y-0)^2+\frac{f'''(0)}{3!}(y-0)^3+\frac{f^{(4)}(0)}{4!}(y-0)^4+\frac{f^{(5)}(0)}{5!}(y-0)^5,$$

com $f(0) = \text{sen}(0) = 0$, $f'(0) = \cos(0) = 1$, $f''(y_0) = -\text{sen}(0) = 0$, $f'''(0) = -\cos(0) = -1$, $f^{(4)}(0) = \text{sen}(0) = 0$ e $f^{(5)}(0) = \cos(0) = 1$.

Ou seja,

$$p_5(y) = y - \frac{y^3}{3!} + \frac{y^5}{5!}.$$

Considere $y = x^4$, segundo a fórmula de Taylor, existe um $\bar{x}$ entre 0 e x^4 tal que: $f(x^4) = p_5(x^4) + E(x^4)$, onde $E(y) = \dfrac{f^{(6)}(\bar{x})}{6!}y^6$. Ou seja,

$$\left|\text{sen}(x^4) - (x^4 - \frac{x^{12}}{3!} + \frac{x^{20}}{5!})\right| = \left|\frac{f^{(6)}(\bar{x})}{6!}(x^4)^6\right| \leq \frac{|x|^{24}}{6!}, \text{ pois } |f^{(6)}(\bar{x})| = |-\text{sen}(\bar{x})| \leq 1$$

(1,0) b) Avalie $\displaystyle\int_0^1 \text{sen}(x^4)dx$ com erro inferior a $\dfrac{1}{2^{12}}$.

$$\left|\int_0^1 \text{sen}(x^4)dx - \int_0^1 p_n(x^4)dx\right| = \left|\int_0^1 \left(\text{sen}(x^4) - p_n(x^4)\right) dx\right| \leq \int_0^1 \left|\text{sen}(x^4) - p_n(x^4)\right| dx$$
$$\leq \int_0^1 \left|f^{(n+1)}(\bar{x})\right| \frac{x^{4n+4}}{(n+1)!}dx \leq \int_0^1 \frac{x^{4n+4}}{(n+1)!}dx = \left[\frac{x^{4n+5}}{(n+1)!(4n+5)}\right]_0^1 = \frac{1}{(n+1)!(4n+5)}$$
$$< \frac{1}{2^{12}} \Longrightarrow n = 5.$$

$\Big($Observe que $\left|f^{(n+1)}(\bar{x})\right| \leq 1$, pois $\left|f^{(n+1)}(\bar{x})\right| = |\cos(\bar{x})|$ ou $\left|f^{(n+1)}(\bar{x})\right| = |\text{sen}(\bar{x})|\Big)$.

$$\int_0^1 \text{sen}(x^4)dx \approx \int_0^1 p_5(x^4)dx = \int_0^1 \left(x^4 - \frac{x^{12}}{3!} + \frac{x^{20}}{5!}\right) dx = \left[\frac{x^5}{5} - \frac{x^{13}}{13.3!} + \frac{x^{21}}{21.5!}\right]_0^1$$
$$= \frac{1}{5} - \frac{1}{13.3!} + \frac{1}{21.5!} = \frac{1}{5} - \frac{1}{78} + \frac{1}{2520} = 0,2 - 0,012821 + 0,000397 = 0,187576.$$