

Calcule as seguintes integrais indefinidas:

(1,5) a) $\int \frac{x^5}{\sqrt{2+x^2}} dx$

Fazendo a mudança de variáveis $x = \sqrt{2} \tan u$, temos que $dx = \sqrt{2} \sec^2 u du$, logo

$$\begin{aligned} \int \frac{x^5}{\sqrt{2+x^2}} dx &= 4\sqrt{2} \int \tan^5 u \sec u du \\ &= 4\sqrt{2} \int (\sec^2 u - 1)^2 (\sec u \tan u) du \\ &= 4\sqrt{2} \int (\sec^4 u - 2\sec^2 u + 1) (\sec u \tan u) du \\ &= 4\sqrt{2} \left(\frac{\sec^5 u}{5} - 2\frac{\sec^3 u}{3} + \sec u \right) + C \\ &= 4\sqrt{2} \left\{ \frac{1}{5} \left(1 + \frac{x^2}{2}\right)^{5/2} - \frac{2}{3} \left(1 + \frac{x^2}{2}\right)^{3/2} + \left(1 + \frac{x^2}{2}\right)^{1/2} \right\} + C \\ &= \frac{1}{5} (2+x^2)^{5/2} - \frac{4}{3} (2+x^2)^{3/2} + 4(2+x^2)^{1/2} + C. \end{aligned}$$

(2,0) b) $\int \frac{6e^x}{(e^x+4)(e^{2x}+4e^x+6)} dx$

Fazendo a mudança de variáveis $e^x = u$, temos que $e^x dx = du$, logo

$$\int \frac{6e^x}{(e^x+4)(e^{2x}+4e^x+6)} dx = \int \frac{6}{(u+4)(u^2+4u+6)} dx. \tag{1}$$

Vamos agora encontrar $A, B, C \in \mathbb{R}$ tais que

$$\frac{6}{(u+4)(u^2+4u+6)} = \frac{A}{u+4} + \frac{Bu+C}{u^2+4u+6}.$$

Isso implica que A, B, C satisfazem o sistema linear

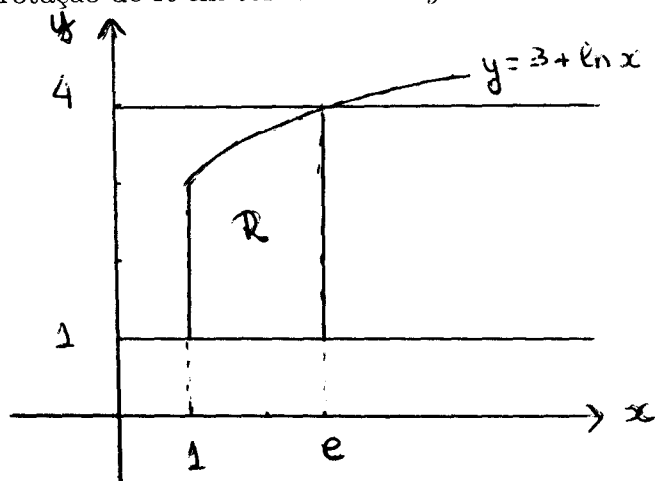
$$\begin{cases} A + B = 0 \\ 4A + 4B + C = 0 \\ 6A + 4C = 6 \end{cases},$$

e portanto, $A = 1, B = -1$ e $C = 0$. Voltando à equação (1), temos que

$$\begin{aligned} \int \frac{6e^x}{(e^x+4)(e^{2x}+4e^x+6)} dx &= \int \left(\frac{1}{u+4} - \frac{u}{u^2+4u+6} \right) du \\ &= \ln|u+4| - \frac{1}{2} \int \frac{u}{\left(\frac{u+2}{\sqrt{2}}\right)^2 + 1} du \\ &= \ln|u+4| - \frac{1}{\sqrt{2}} \int \frac{\sqrt{2}v-2}{v^2+1} dv, \text{ onde } v = \frac{u+2}{\sqrt{2}} \\ &= \ln|u+4| - \frac{1}{2} \ln(1+v^2) + \sqrt{2} \arctan v + C \\ &= \ln(e^x+4) - \frac{1}{2} \ln \left(1 + \left(\frac{e^x+2}{\sqrt{2}} \right)^2 \right) + \sqrt{2} \arctan \left(\frac{e^x+2}{\sqrt{2}} \right) + C \end{aligned}$$

Questão 2. (2,5)

Seja $R = \{(x, y) \in \mathbb{R}^2 : 1 \leq x \leq e, 1 \leq y \leq 3 + \ln(x)\}$. Calcule o volume do sólido obtido pela rotação de R em torno da reta $y = 4$.



$$V = \pi \int_1^e [(4-1)^2 - (4-3-\ln x)^2] dx = \pi \int_1^e [8 + 2\ln x - \ln^2 x] dx =$$

$$= \pi \left[8(e-1) + 2 \int_1^e \ln x dx - \int_1^e \ln^2 x dx \right]$$

Cálculo das integrais (per partes)

$$\int_1^e 1 \ln x dx = x \ln x \Big|_1^e - \int_1^e x \frac{1}{x} dx = (x \ln x - x) \Big|_1^e =$$

$$= (e \ln e) - (1 \ln 1 - 1) = 1$$

$$\int_1^e 1 \ln^2 x dx = x \ln^2 x \Big|_1^e - \int_1^e x (2 \ln x) \frac{1}{x} dx = x \ln^2 x \Big|_1^e - 2 \int_1^e \ln x dx =$$

$$= (e \ln^2 e - 1 \ln^2 1) - 2 = e - 2$$

$$\text{Resposta: } V = \pi [8(e-1) + 2 - (e-2)] = \pi (7e-4)$$

3. Considere a função $g(x) = \int_0^{2\text{sen}(x)} [3 + \cos(t^2)] dt$ definida em \mathbb{R} .

(1,0) a) Calcule $g'(x)$.

(1,0) b) Seja $f(x) = \int_0^{g(x)} \frac{x^2}{\sqrt{4+t^4}} dt$. Calcule $f'(\pi)$.

Solução:

a) Usando o segundo teorema fundamental do Cálculo, obtemos

$$\begin{aligned} g'(x) &= \frac{d}{dx} \left(\int_0^{2\text{sen}(x)} [3 + \cos(t^2)] dt \right) \\ &= [3 + \cos(4\text{sen}^2(x))] (2\cos(x)) \end{aligned}$$

b) Novamente, pelo segundo teorema fundamental do Cálculo

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left(\int_0^{g(x)} \frac{x^2}{\sqrt{4+t^4}} dt \right) \\ &= \frac{d}{dx} \left(x^2 \int_0^{g(x)} \frac{1}{\sqrt{4+t^4}} dt \right) \\ &= 2x \int_0^{g(x)} \frac{1}{\sqrt{4+t^4}} dt + x^2 \left(\frac{1}{\sqrt{4+(g(x))^4}} \right) g'(x) \end{aligned}$$

Fazendo $x = \pi$ na expressão acima e observando que $g(\pi) = 0$, obtemos

$$f'(\pi) = \pi^2 \cdot \frac{1}{2} \cdot (-8) = -4\pi^2$$

Questão 4.

(1,0) a) Mostre que para todo $x \in \mathbb{R}$ temos

$$\left| \text{sen}(x^4) - \left(x^4 - \frac{x^{12}}{3!} + \frac{x^{20}}{5!} \right) \right| \leq \frac{|x|^{24}}{6!}$$

Sejam $f(y) = \text{sen}(y)$ e $y_0 = 0$.

O polinômio de Taylor de f de grau 5, em torno de y_0 é dado por:

$$p_5(y) = f(0) + f'(0)(y-0) + \frac{f''(0)}{2!}(y-0)^2 + \frac{f'''(0)}{3!}(y-0)^3 + \frac{f^{(4)}(0)}{4!}(y-0)^4 + \frac{f^{(5)}(0)}{5!}(y-0)^5,$$

com $f(0) = \text{sen}(0) = 0$, $f'(0) = \text{cos}(0) = 1$, $f''(y_0) = -\text{sen}(0) = 0$, $f'''(0) = -\text{cos}(0) = -1$, $f^{(4)}(0) = \text{sen}(0) = 0$ e $f^{(5)}(0) = \text{cos}(0) = 1$.

Ou seja,

$$p_5(y) = y - \frac{y^3}{3!} + \frac{y^5}{5!}.$$

Considere $y = x^4$, segundo a fórmula de Taylor, existe um \bar{x} entre 0 e x^4 tal que:

$f(x^4) = p_5(x^4) + E(x^4)$, onde $E(y) = \frac{f^{(6)}(\bar{x})}{6!}y^6$. Ou seja,

$$\left| \text{sen}(x^4) - \left(x^4 - \frac{x^{12}}{3!} + \frac{x^{20}}{5!} \right) \right| = \left| \frac{f^{(6)}(\bar{x})}{6!}(x^4)^6 \right| \leq \frac{|x|^{24}}{6!}, \text{ pois } |f^{(6)}(\bar{x})| = |-\text{sen}(\bar{x})| \leq 1$$

(1,0) b) Avalie $\int_0^1 \text{sen}(x^4) dx$ com erro inferior a $\frac{1}{2^{12}}$.

$$\begin{aligned} \left| \int_0^1 \text{sen}(x^4) dx - \int_0^1 p_n(x^4) dx \right| &= \left| \int_0^1 (\text{sen}(x^4) - p_n(x^4)) dx \right| \leq \int_0^1 |\text{sen}(x^4) - p_n(x^4)| dx \\ &\leq \int_0^1 |f^{(n+1)}(\bar{x})| \frac{x^{4n+4}}{(n+1)!} dx \leq \int_0^1 \frac{x^{4n+4}}{(n+1)!} dx = \left[\frac{x^{4n+5}}{(n+1)!(4n+5)} \right]_0^1 = \frac{1}{(n+1)!(4n+5)} \\ &< \frac{1}{2^{12}} \implies n = 5. \end{aligned}$$

(Observe que $|f^{(n+1)}(\bar{x})| \leq 1$, pois $|f^{(n+1)}(\bar{x})| = |\text{cos}(\bar{x})|$ ou $|f^{(n+1)}(\bar{x})| = |\text{sen}(\bar{x})|$).

$$\begin{aligned} \int_0^1 \text{sen}(x^4) dx &\approx \int_0^1 p_5(x^4) dx = \int_0^1 \left(x^4 - \frac{x^{12}}{3!} + \frac{x^{20}}{5!} \right) dx = \left[\frac{x^5}{5} - \frac{x^{13}}{13 \cdot 3!} + \frac{x^{21}}{21 \cdot 5!} \right]_0^1 \\ &= \frac{1}{5} - \frac{1}{13 \cdot 3!} + \frac{1}{21 \cdot 5!} = \frac{1}{5} - \frac{1}{78} + \frac{1}{2520} = 0,2 - 0,012821 + 0,000397 = 0,187576. \end{aligned}$$