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18.01 Single Variable Calculus
Fall 2006

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Problem 1. (20 points) Evaluate the following integrals

10pts a) $\int_0^2 \frac{x dx}{(1+x^2)^2} = \frac{1}{2} \int_1^5 \frac{du}{u^2} = -\frac{1}{2u} \Big|_1^5$
 $u = 1+x^2$
 $du = 2x dx$
 $= -\frac{1}{10} - (-\frac{1}{2}) =$
 $= \frac{1}{2} - \frac{1}{10} = \boxed{\frac{2}{5}}$

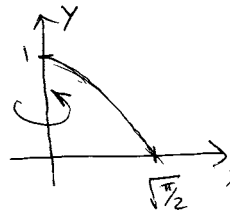
10pts b) $\int_{-\pi/2}^{\pi/2} \sin^6 x \cos x dx = \int_{-1}^1 u^6 du = \frac{1}{7} u^7 \Big|_{-1}^1$
 $u = \sin x$
 $du = \cos x dx$
 $= \boxed{\frac{2}{7}}$

Problem 3. (20 points) Find the volume of the solid of revolution formed by revolving around the y -axis the region enclosed by

$$y = \cos(x^2)$$

and the x -axis (central hump, only).

$u = x^2$
 $du = 2x dx$

$$V = \int_0^{\sqrt{\pi/2}} 2\pi x \cdot \cos(x^2) dx = \pi \int_0^{\sqrt{\pi/2}} \cos u du = \pi \sin u \Big|_0^{\sqrt{\pi/2}} = \boxed{\pi}$$


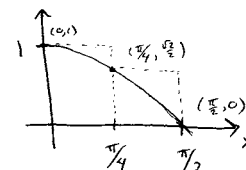
Problem 2. (20 points) Find the following approximations to

$$\int_0^{\pi/2} \cos x dx$$

(Do not give a numerical approximation to square roots; leave them alone.)

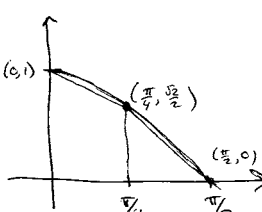
a) using the upper Riemann sum with two intervals

$\int_0^{\pi/2} \cos x dx \approx \frac{\pi}{4} \cdot 1 + \frac{\pi}{4} \cos \frac{\pi}{4}$
 $= \frac{\pi}{4} (1 + \frac{\sqrt{2}}{2})$
 (≈ 1.341)



b) using the trapezoidal rule with two intervals

$\int_0^{\pi/2} \cos x dx \approx \frac{1}{2} \frac{\pi}{4} (1 + \frac{\sqrt{2}}{2}) + \frac{1}{2} \frac{\pi}{4} \cdot \frac{\sqrt{2}}{2}$
 $= \frac{\pi}{8} (1 + \sqrt{2})$
 (≈ 0.948)



c) using Simpson's rule with two intervals

$\int_0^{\pi/2} \cos x dx \approx \frac{1}{3} \frac{\pi}{4} (1 + 4 \cos \frac{\pi}{4} + 0)$
 $= \frac{\pi}{12} (1 + 2\sqrt{2})$
 (≈ 1.002)

Problem 4. (20 points) Students studying for an exam get x hours of sleep in the two days leading up to the exam, where x is in the range $0 \leq x \leq a$. The number of students who got between x_1 and x_2 hours of sleep is given by

$$\int_{x_1}^{x_2} cx dx; \quad 0 \leq x_1 \leq x_2 \leq a$$

10pts a) What fraction of the students got less than $a/2$ hours of sleep?

Total number of students = $\int_0^a cx dx = \frac{ca^2}{2}$ (all students get between 0 and a hours of sleep).
 number of students who got between 0 and $a/2$ hours of sleep = $\int_0^{a/2} cx dx = c \frac{a^2}{8}$
 ratio = $\frac{ca^2/8}{ca^2/2} = \boxed{\frac{1}{4}}$

10pts b) Their scores are proportional to the amount of sleep they got: $S(x) = 100(x/a)$. Find the (correctly weighted) average score in the class.

$N = \text{Total number of students} = \int_0^a cx dx = \frac{ca^2}{2}$
 Average score = $\frac{1}{N} \int_0^a cx S(x) dx = \frac{1}{N} \int_0^a \frac{100c}{a} x^2 dx$
 $= \frac{1}{ca^2/2} \cdot \frac{100c}{a} \cdot \frac{a^3}{3} = \boxed{\frac{200}{3}} (\approx 66.6)$

Problem 5. (20 points) Let

$$F(x) = \int_0^x \sqrt{t} \sin t \, dt$$

5 pts a) Find $F'(x)$ for $x > 0$ and identify the points $a > 0$ where $F'(a) = 0$.

$$F'(x) = \sqrt{x} \sin x$$

$$F'(a) = 0, a > 0, \text{ for } a = k\pi \quad k \text{ a positive integer. } (1, 2, 3, 4, \dots)$$

5 pts b) Decide whether F has a local maximum or minimum at the smallest critical point $a > 0$ that you found in part (a) by evaluating F'' .

The smallest ^{positive} critical point is at $a = \pi$.

$$F''(x) = \frac{1}{2\sqrt{x}} \sin x + \sqrt{x} \cos x$$

$$F''(\pi) = \frac{1}{2\sqrt{\pi}} \underbrace{\sin \pi}_0 + \sqrt{\pi} \underbrace{\cos \pi}_{-1} = -\sqrt{\pi} < 0$$

So π is a local maximum.

5 pts c) Say whether $F(x)$ is positive, negative or zero at each of the following points, and give a reason in each case.

1 pt i) $x = 0$ $F(0) = \int_0^0 \sqrt{t} \sin t \, dt = 0$ since the interval of integration has length 0.

2 pts ii) $x = \pi$ $F(\pi) = \int_0^\pi \sqrt{t} \sin t \, dt > 0$ since for t between 0 and π the integrand, $\sqrt{t} \sin t$, is positive.

2 pts iii) $x = 2\pi$ $F(2\pi) = \int_0^{2\pi} \sqrt{t} \sin t \, dt = \int_0^\pi \sqrt{t} \sin t \, dt + \int_\pi^{2\pi} \sqrt{t} \sin t \, dt$
 $= \int_0^\pi \sqrt{t} \sin t \, dt - \int_\pi^{2\pi} \sqrt{t} |\sin t| \, dt < 0$ since $\sqrt{t} |\sin t| < \sqrt{t+\pi} |\sin(t+\pi)|$.

5 pts d) Use a change of variable to express $G(x) = \int_0^x u^2 \sin(u^2) \, du$ in terms of F .

Let $t = u^2, dt = 2u \, du$

$$G(x) = \int_0^x u^2 \sin(u^2) \, du = \int_0^{x^2} t \sin t \frac{dt}{2\sqrt{t}}$$

$$= \frac{1}{2} \int_0^{x^2} \sqrt{t} \sin t \, dt = \frac{1}{2} F(x^2)$$

In other words, "there is more negative area between π and 2π than positive area from 0 to π ."