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18.01 Single Variable Calculus
Fall 2006

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Problem 1. (15 points) Evaluate $\int \frac{dx}{x(x+1)^2}$

$$\frac{1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

"Cover-up" method yields $\boxed{A=1}$ $(x=0)$ $\boxed{C=-1}$ $(x=-1)$

Let $x=1$:

$$\frac{1}{4} = \frac{1}{1} + \frac{B}{2} + \frac{-1}{4} \Rightarrow \boxed{B=-1}$$

$$\int \frac{dx}{x(x+1)^2} = \int \left[\frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2}\right] dx$$

$$= \boxed{\ln|x| - \ln|x+1| + \frac{1}{x+1} + \text{const.}}$$

(Also ok: $\ln x - \ln(x+1) + \frac{1}{x+1} + \text{const.}$)

Problem 2. (15 points) Evaluate $\int (\ln x)x^2\,dx$

$$\int \underbrace{\ln x}_{u}\ \underbrace{x^2}_{v'}\ dx = (\ln x)\frac{x^3}{3} - \int \frac{x^3}{3}\cdot\frac{1}{x}\,dx$$

$x^2/3$

$$\left[u' = \frac{1}{x},\ v = \frac{x^3}{3}\right] \qquad = \boxed{\frac{x^3}{3}\ln x - \frac{x^3}{9} + C}$$

Problem 3. (20 points) Use a trigonometric subsititution to evaluate $\int_0^1 \frac{dx}{(4+x^2)^{3/2}}$. (Be careful evaluating the limits.)

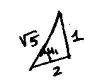
$x = 2\tan u$, $dx = 2\sec^2 u\,du$

$$\int_0^1 \frac{dx}{(4+x^2)^{3/2}} = \int_{u_0}^{u_1} \frac{2\sec^2 du}{(4+4\tan^2 u)^{3/2}} \qquad \left(\begin{matrix}2\tan u_1 = 1\\ 2\tan u_0 = 0\\ \Rightarrow u_0 = 0.\end{matrix}\right)$$

$$= \int_0^{u_1} \frac{2\sec^2 u\,du}{8\sec^3 u} = \frac{1}{4}\int_0^{u_1} \cos u\,du$$

$$= \frac{1}{4}\sin u_1 = \boxed{\frac{1}{4}\cdot\frac{1}{\sqrt{5}}}$$

$2\tan u_1 = 1 \Rightarrow \tan u_1 = 1/2$



$\boxed{\sin u_1 = 1/\sqrt{5}}$

Problem 4. a. (10 points) Find an integral formula for the arclength of the curve $y = 2\sqrt{x+1}$ for $0 \le x \le 1$. Do not evaluate.

$$y' = \frac{1}{\sqrt{x+1}}$$

$$\text{Length} = \int_0^1 \underbrace{\sqrt{1+(y')^2}\,dx}_{ds} = \boxed{\int_0^1 \sqrt{1+\frac{1}{x+1}}\ dx}$$

b. (10 points) Find an integral formula for the surface area of the curve in part (a) rotated around the x-axis. Simplify the integrand, and evaluate the integral.

$$\text{AREA} = \int_0^1 2\pi y\,ds = \int_0^1 2\pi\,(2\sqrt{x+1})\sqrt{1+\frac{1}{x+1}}\ dx$$

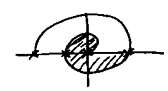
$$= 4\pi \int_0^1 \sqrt{(x+1)+1}\ dx$$

$$= 4\pi \int_0^1 (x+2)^{1/2}\,dx$$

$$= 4\pi\,\frac{2}{3}(x+2)^{3/2}\Big|_0^1$$

$$= \boxed{\frac{8\pi}{3}\left(3^{3/2} - 2^{3/2}\right)}$$

Problem 5. a. (7 points) Sketch the spiral $r = \theta^2$, $0 \le \theta \le 3\pi$. Say how many times the curve meets the x-axis counting $\theta = 0$ as the first time, and mark those points with X-s. (Your sketch need not be accurate to scale.)



MEETS OR CROSSES 4 TIMES

b. (8 points) On your picture, shade in the region $0 \le r \le \theta^2$, $0 \le \theta \le 2\pi$, and find its area.

$$\text{AREA} = \int_0^{2\pi} \frac{1}{2}r^2\,d\theta = \int_0^{2\pi} \frac{1}{2}\theta^4\,d\theta = \frac{1}{10}\theta^5\Big|_0^{2\pi}$$

$$= \boxed{\frac{(2\pi)^5}{10}}$$

or $\frac{16\pi^5}{5}$

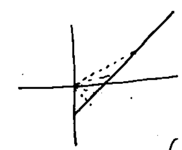
Problem 6. a. (10 points) Find the equation in polar coordinates for the line $y = x - 1$ in the form $r = f(\theta)$.

$r\sin\theta = r\cos\theta - 1$

$\Leftrightarrow\quad 1 = r\cos\theta - r\sin\theta$

$\Leftrightarrow\quad \boxed{r = \frac{1}{\cos\theta - \sin\theta}}$

b. (5 points) Find the range of θ for the portion of the line $y = x - 1$ in the range $0 \le x < \infty$. (It helps to draw a picture.)



$\boxed{-\pi/2 \le \theta < \pi/4}$

($-\infty \le$ slope of dotted rays from origin to $y=x-1$ < 1)