18.01 Exam 5

Problem 1(25 points) Use integration by parts to compute the antiderivative,

$$\int x^3 e^{-x^2} dx.$$

Solution to Problem 1 The derivative of e^{-x^2} is $-2xe^{-x^2}$. Thus, set

$$\begin{array}{ll} u = x^2, & dv = xe^{-x^2}dx \\ du = 2xdx & v = -e^{-x^2}/2. \end{array}$$

Then, by integration by parts,

$$\int udv = uv - \int vdu,$$

$$\int x^3 e^{-x^2} dx = -\frac{1}{2}x^2 e^{-x^2} + \int xe^{-x^2} dx.$$

As computed above, the derivative of e^{-x^2} is $-2xe^{-x^2}$. Thus the new integral is $-e^{-x^2}/2 + C$. Therefore,

$$\int x^3 e^{-x^2} dx = -\frac{1}{2}x^2 e^{-x^2} - \frac{1}{2}e^{-x^2} = \boxed{-(x^2+1)e^{-x^2}/2.}$$

Problem 2(20 points) Use polynomial division and factoring to compute the antiderivative,

$$\int \frac{x^3 - 1}{x^2 - 3x + 2} dx.$$

You will not need to use partial fractions (though you are free to do so).

Solution to Problem 2 By the polynomial division algorithm,

$$x^3 - 1 = (x+3)(x^2 - 3x + 2) + (7x - 7).$$

Thus the fraction is,

$$\frac{x^3 - 1}{x^2 - 3x + 2} = x + 3 + \frac{7(x-1)}{x^2 - 3x + 2}.$$

The denominator of the new fraction factors as,

$$x^2 - 3x + 2 = (x-1)(x-2).$$

Thus the original fraction is,

$$x + 3 + \frac{7(x-1)}{(x-1)(x-2)} = x + 3 + \frac{7}{x-2}.$$

Therefore the antiderivative is,

$$\int x + 3 + \frac{7}{x-2} dx = \boxed{(1/2)x^2 + 3x + 7\ln(|x-2|) + C.}$$

Problem 3(20 points)

(a)(15 points) Find the partial fraction decomposition of,

$$\frac{x+3}{x^2 - 2x + 1}.$$

Solution to (a) Using the quadratic formula, $x^2 - 2x + 1$ has only the root 1. Thus it is,

$$x^2 - 2x + 1 = (x-1)^2.$$

So the partial fraction decomposition has the form,

$$\frac{x+3}{x^2 - 2x + 1} = \frac{A}{(x-1)^2} + \frac{B}{x-1},$$

for some choice of A and B. By the Heaviside cover-up method, A equals,

$$(x + 3|_{x=1} = 4.$$

Thus the partial fraction decomposition is,

$$\frac{x+3}{x^2 - 2x + 1} = \frac{4}{(x-1)^2} + \frac{B}{x-1}.$$

Plugging in $x = 0$ gives,

$$3 = \frac{x+3}{x^2 - 2x + 1}|_{x=0} = \frac{4}{(0-1)^2} + \frac{B}{0-1} = 4 - B.$$

Therefore B equals 1. So the partial fraction decomposition is,

$$\frac{x+3}{x^2 - 2x + 1} = \boxed{4/(x-1)^2 + 1/(x-1).}$$

(b)(5 points) Use your answer from (a) to compute the antiderivative of

$$\int \frac{x+3}{x^2-2x+1}dx.$$

Solution to (b) By the **Solution to (a)**, the integral is,

$$\int \frac{x+3}{x^2-2x+1}dx = \int \frac{4}{(x-1)^2} + \frac{1}{x-1}dx.$$

The second integral is easily computed and gives,

$$\int \frac{x+3}{x^2-2x+1}dx = \boxed{-4/(x-1) + \ln(|x-1|) + C.}$$

Problem 4(25 points) Let a be a positive real number.

(a)(3 points) Determine the range of x on which $2ax - x^2$ is nonnegative.

Solution to (a) Completing the square gives,

$$-x^2 + 2ax = -(x-a)^2 + a^2.$$

Therefore the expression is nonnegative when,

$$-(x-a)^2 + a^2 \geq 0 \Leftrightarrow a^2 \geq (x-a)^2.$$

Taking square roots, the expression is nonnegative when,

$$a \geq x - a \geq -a.$$

This simplifies to,

$$\boxed{0 \leq x \leq 2a.}$$

(b)(22 points) For that range, compute the antiderivative,

$$\int \sqrt{2ax - x^2}dx.$$

Solution to (b) Begin by making the linear change of variables,

$$u = x - a, \quad du = dx.$$

The new integral is,

$$\int \sqrt{a^2 - u^2}du.$$

This integral can be computed using an inverse trigonometric substitution,

$$u = a\sin(\theta), \quad du = a\cos(\theta)d\theta.$$

Plugging in, the new integral is,

$$\int \sqrt{a^2(1-\sin^2(\theta))}(a\cos(\theta)d\theta) = a^2 \int \cos^2(\theta)d\theta.$$

This can be solved either using integration by parts or using the half-angle formula from trigonometry. The half-angle formula is decidedly faster and gives,

$$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}.$$

Substituting this in, the new integral is,

$$a^2 \int \frac{1}{2} + \frac{1}{2}\cos(2\theta)d\theta = a^2\left(\frac{\theta}{2} + \frac{1}{4}\sin(2\theta)\right) + C.$$

Using the double-angle formula, this simplifies to,

$$a^2\left(\frac{\theta}{2} + \frac{1}{2}\sin(\theta)\cos(\theta)\right) + C.$$

Back-substituting for u gives,

$$\frac{a^2}{2}\sin^{-1}(u/a) + \frac{1}{2}u\sqrt{a^2-u^2} + C.$$

Back-substituting for x gives,

$$\int \sqrt{2ax-x^2}dx = \boxed{(a^2\sin^{-1}(u/a) + (x-a)\sqrt{2ax-x^2})/2 + C.}$$

Problem 5(10 points) Compute the following derivatives. Please show all work to receive full credit.

(a)(5 points)

$$y = \sinh(t) = \frac{e^t - e^{-t}}{2}, \quad \frac{dy}{dx} = ?$$

Solution to (a) The derivative should be with respect to t. Using the chain rule, it is,

$$\frac{d}{dt}\left(\frac{e^t}{2} - \frac{e^{-t}}{2}\right) = \left(\frac{1}{2}e^t - \frac{1}{2}(-e^{-t})\right) = \boxed{\cosh(t).}$$

(b)(5 points)

$$y = \sinh^{-1}(x) = \ln(x + \sqrt{x^2+1}), \quad \frac{dy}{dx} = ?$$

There are at least 2 methods. You may use the one you prefer.

Solution to (b) The formula for the derivative of an inverse function is,

$$\frac{df^{-1}}{dx}(x) = \frac{1}{f'(f^{-1}(x))}.$$

As computed above, the derivative of $\sinh(x)$ is $\cosh(x)$. Thus the derivative is,

$$\frac{1}{\cosh(\sinh^{-1}(x))}.$$

Using the identity $\cosh^2(x) - \sinh^2(x) = 1$, the denominator is,

$$\cosh(\sinh^{-1}(x)) = \sqrt{1 + [\sinh(\sinh^{-1}(x))]^2} = \sqrt{1+x^2}.$$

Therefore the derivative is,

$$\frac{d}{dx}\sinh^{-1}(x) = \boxed{1/\sqrt{1+x^2}.}$$

Extra credit(5 points)

$$y = \sin(\tan^{-1}(2x)), \quad \frac{dy}{dx} = ?$$

Only solutions in simplest terms will be accepted.

Solution to the extra credit problem Denote by θ the angle $\tan^{-1}(2x)$. There is a right triangle with angle θ having opposite side $2x$ and adjacent side 1. Therefore the hypotenuse has length,

$$\sqrt{(2x)^2 + (1)^2} = \sqrt{4x^2+1}.$$

Since $\sin(\theta)$ is the ratio of the opposite side by the hypotenuse,

$$\sin(\tan^{-1}(2x)) = \frac{2x}{\sqrt{4x^2+1}}.$$

Using the quotient rule and the chain rule,

$$\frac{d}{dx}\left(\frac{2x}{\sqrt{4x^2+1}}\right) = \frac{1}{4x^2+1}\left((2)\sqrt{4x^2+1} - (2x)\left(\frac{1}{2\sqrt{4x^2+1}}(8x)\right)\right) = \frac{2}{(4x^2+1)^{3/2}}((4x^2+1)-4x^2).$$

Therefore the derivative is,

$$\frac{d}{dx}\sin(\tan^{-1}(2x)) = \boxed{2/(4x^2+1)^{3/2}.}$$