18.01 Single Variable Calculus
Fall 2006

Problem 1. (10 pts). Find the tangent line to $y = \frac{1}{3}x^2$ at $x = 1$.

$f(x) = \frac{1}{3}x^2 \quad , f'(1) = \frac{2}{3}x\Big|_{x=1} = \frac{2}{3}$

$x_0 = 1, \quad y_0 = f(x_0) = \frac{1}{3}x_0^2 = 1/3$

$y - y_0 = f'(1)(x - x_0)$

$y - \frac{1}{3} = \frac{2}{3}(x - 1)$

$$\boxed{y = \frac{2}{3}x - \frac{1}{3}}$$

Problem 2. Find the derivative of the following functions:

a. (7 pts). $\dfrac{x}{\sqrt{1-x}}$ $\quad x < 1$

$$f'(x) = \frac{\sqrt{1-x} - x\frac{-1}{2\sqrt{1-x}}}{1-x} = \frac{1-x+\frac{x}{2}}{(1-x)^{3/2}} \boxed{= \frac{2-x}{2(1-x)^{3/2}}}$$

b. (8 pts). $\dfrac{\cos(2x)}{x}$

$$f'(x) = \boxed{\frac{-2\sin(2x)\cdot x - \cos(2x)}{x^2}}$$

c. (5 pts). $e^{2f(x)} = g(x)$

$$g'(x) = \boxed{2f'(x)\,e^{2f(x)}}$$

d. (5 pts). $\ln(\sin x)$

$$f'(x) = \frac{1}{\sin x}\cdot\cos x = \boxed{\cot x}$$

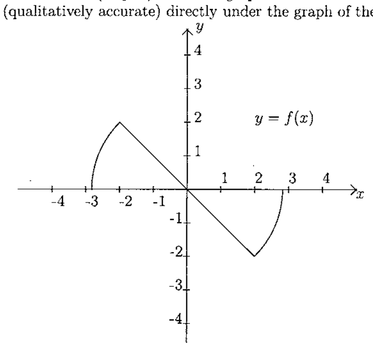
Problem 3. (15 pts). Find $\frac{dy}{dx}$ for the function y defined implicitly by $y^4 + xy = 4$ at $x = 3$, $y = 1$.

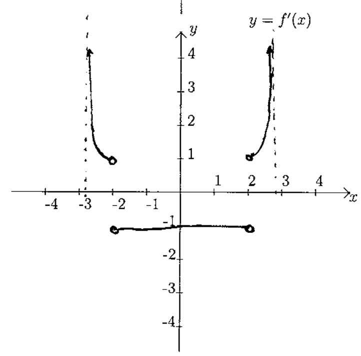
$4y^3\frac{dy}{dx} + y + x\frac{dy}{dx} = 0$

$\dfrac{dy}{dx} = \dfrac{-y}{4y^3 + x}$

$$\frac{dy}{dx}\Big|_{x=3,\,y=1} = \frac{-1}{4(1)^3 + 3} = \boxed{-1/7}$$

Problem 4. (15 pts.) Draw the graph of the derivative of the function (qualitatively accurate) directly under the graph of the function.





Problem 5. (15 pts) Let

$$f(x) = \begin{cases} ax + b & x < 1 \\ x^4 + x + 1 & x \geq 1 \end{cases}$$

Find all a and b such that the function $f(x)$ is differentiable.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (ax+b) = a+b$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^4 + x + 1) = 3$$

So $f(x)$ is continuous at $x=1$ if and only if $a+b=3$

$$\lim_{x \to 1^-} f'(x) = \lim_{x \to 1^-} a = a$$

$$\lim_{x \to 1^+} f'(x) = \lim_{x \to 1^+} (4x^3+1) = 5$$

So $f'(x)$ is differentiable at $x=1$ if and only if $\boxed{a=5.}$ So $\boxed{b=-2}$

Problem 6. Evaluate these limits by relating them to a derivative.

a. (5 pts). Evaluate $\lim_{x \to 0} \frac{(1+2x)^{10} - 1}{x}$.

Let $f(x) = (1+2x)^{10}$. Then

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{(1+2h)^{10} - 1}{h}$$

$$\| $$

$$10(1+2x)^9 \cdot 2 \Big|_{x=0} = \boxed{20}$$

b. (5 pts). Evaluate $\lim_{x \to 0} \frac{\sqrt{\cos x} - 1}{x}$.

Let $f(x) = \sqrt{\cos x}$. Then

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{\sqrt{\cos h} - 1}{h}$$

So $$\lim_{h \to 0} \frac{\sqrt{\cos h} - 1}{h} = f'(0) = \frac{1}{2}(\cos x)^{-1/2}(-\sin x)\Big|_{x=0}$$

$$= \frac{1}{2} \frac{1}{\sqrt{1}} \cdot (-0) = \boxed{0}$$

Problem 7. (10 pts). Derive the formula $\frac{d}{dx} a^x = M(a)a^x$ directly from the definition of the derivative, and identify $M(a)$ as a limit.

$$\frac{d}{dx}(a^x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h \to 0} \frac{a^x a^h - a^x}{h}$$

$$= \lim_{h \to 0} \frac{a^x(a^h - 1)}{h}$$

$$= a^x \underbrace{\lim_{h \to 0} \frac{a^h - 1}{h}}_{M(a)}$$