3ª Prova de MA211/A/B (12/11/2010)

RA: _______ Nome: GABARITO

Turma: ___

Questão	Nota
1	
2	
3	
4	
Total	

1. (2,0 pontos) Calcule a integral $\int_0^1 \int_{x^2}^1 x^3 sen(y^3) dydx$. (**Sugestão:** inverta a ordem de integração).

2. (3,0 pontos) Calcule a integral $\iint_B \cos\left(\dfrac{y-x}{y+x}\right) dxdy$ onde B é a região trapezoidal com vértices (1, 0), (2, 0), (0, 2) e (0, 1), usando uma mudança de variáveis conveniente.

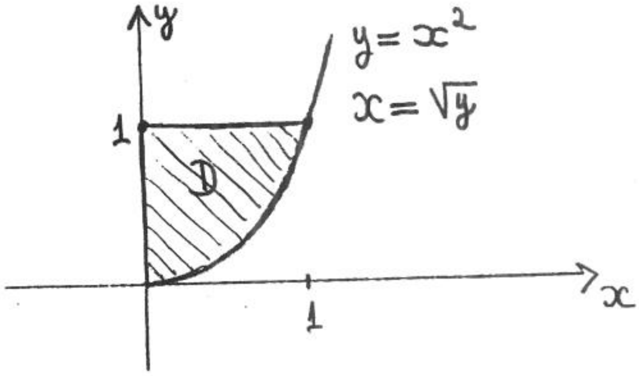
3. (2,5 pontos) Calcule o volume do sólido formado pelos pontos $(x,\ y,\ z) \in \Re^3$ que satisfazem $x^2 + y^2 + z^2 \leq 9$ e $z \geq 2\sqrt{x^2 + y^2}$. (**Sugestão:** use coordenadas esféricas).

4. (2,5 pontos) Considere o campo de força $\vec{F}(x,y,z) = y^2\vec{i} + x\vec{j} - z\vec{k}$, e seja γ a poligonal de vértices $A = (0,\ 1,\ -1)$, $B = (1,\ 1,\ 0)$ e $C = (1,\ 2,\ -1)$, orientada de A para C. Determine o trabalho realizado por $\vec{F}$ para deslocar uma partícula de A até C, ao longo da poligonal.

1. Calcule a integral $\int_0^1 \int_{x^2}^1 x^3 \operatorname{sen}(y^3)\, dy\, dx$.

(DICA: inverta a ordem de integração)(ex. 15.3.42).

SOLUÇÃO:



$$D: \begin{cases} 0 \leqslant x \leqslant 1 \\ x^2 \leqslant y \leqslant 1 \end{cases}$$

$$D: \begin{cases} 0 \leqslant y \leqslant 1 \\ 0 \leqslant x \leqslant \sqrt{y} \end{cases}$$

$$\int_0^1 \int_{x^2}^1 x^3 \operatorname{sen}(y^3)\, dy\, dx = \iint_D x^3 \operatorname{sen}(y^3)\, dA$$

$$\overset{1,2}{=} \int_0^1 \int_0^{\sqrt{y}} x^3 \operatorname{sen}(y^3)\, dx\, dy$$

$$\begin{cases} u = y^3 \\ du = 3y^2 dy \end{cases} \quad = \int_0^1 \left[\frac{x^4}{4} \operatorname{sen}(y^3) \Big|_{x=0}^{x=\sqrt{y}} \right] dy$$

$$\overset{0,2}{=} \frac{1}{4} \int_0^1 y^2 \operatorname{sen}(y^3)\, dy$$

$$\overset{0,4}{=} \frac{1}{12} \int_0^1 \operatorname{sen} u\, du = -\frac{1}{12} \cos u \Big|_{u=0}^{u=1}$$

$$\overset{0,2}{=} \frac{1}{12}(1 - \cos 1)$$

QUESTÃO 2: $I = \iint_R \cos\left(\frac{y-x}{y+x}\right) dA$

R = região trapezoidal com vértices $(1,0)$, $(2,0)$, $(0,2)$ e $(0,1)$

$$\begin{cases} u = y - x \\ v = y + x \end{cases} \Rightarrow u + v = 2y \Rightarrow y = \frac{u+v}{2}, \quad x = y - u = \frac{u+v}{2} - \frac{2u}{2} = \frac{v-u}{2}$$

0,5

$$\begin{cases} x = \frac{v-u}{2} \\ y = \frac{u+v}{2} \end{cases} \qquad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

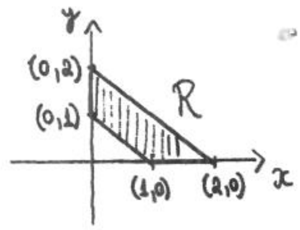
0,3

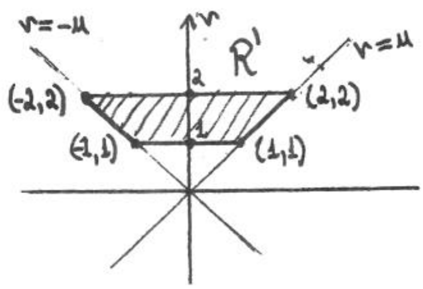
$(x,y) = (1,0) \Rightarrow (u,v) = (-1,1)$

$(x,y) = (2,0) \Rightarrow (u,v) = (-2,2)$

$(x,y) = (0,1) \Rightarrow (u,v) = (1,1)$

$(x,y) = (0,2) \Rightarrow (u,v) = (2,2)$





1,0

$$I = \iint_{R'} \left(\cos \frac{u}{v}\right) \frac{1}{2}\, du\, dv$$

$$= \frac{1}{2} \int_1^2 \int_{-v}^{v} \cos\frac{u}{v}\, du\, dv =$$

0,6

$$= \frac{1}{2} \int_1^2 \left(v \cdot \operatorname{sen}\frac{u}{v} \Big|_{u=-v}^{u=v} \right) dv = \frac{1}{2} \int_1^2 \left(v \operatorname{sen} 1 - v \operatorname{sen}(-1) \right) dv$$

$$= \cancel{\frac{1}{2}}\, \cancel{2} \operatorname{sen} 1 \int_1^2 v\, dv = (\operatorname{sen} 1) \frac{v^2}{2} \Big|_1^2 = (\operatorname{sen} 1)\left(\frac{4}{2} - \frac{1}{2}\right) = \frac{3}{2} \operatorname{sen} 1$$

0,6

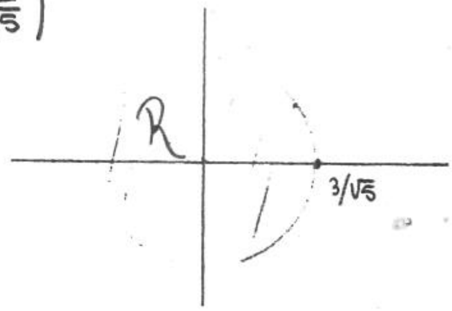
3,0

3) Calcule o volume do sólido formado pelos pontos $(x,y,z) \in \mathbb{R}^3$ que satisfazem $x^2+y^2+z^2 \leqslant 9$ e $z \geqslant 2\sqrt{x^2+y^2}$.

SOLUÇÃO $\begin{cases} x^2+y^2+z^2 = 9 \\ z^2 = 4(x^2+y^2) \end{cases} \Rightarrow 9 = x^2+y^2+(4x^2+4y^2) = 5x^2+5y^2$

$\Rightarrow x^2+y^2 = \left(\dfrac{3}{\sqrt{5}}\right)^2$

$\begin{cases} x = r\cos\theta \\ y = r\,\mathrm{sen}\,\theta \\ r = \sqrt{x^2+y^2} \end{cases} \qquad R': \begin{cases} 0 \leqslant \theta \leqslant 2\pi \\ 0 \leqslant r \leqslant \frac{3}{\sqrt{5}} \end{cases}$



$$2\sqrt{x^2+y^2} \leqslant z \leqslant \sqrt{9-x^2-y^2}$$

$$2r \leqslant z \leqslant \sqrt{9-r^2}$$

1,0

$$\text{Volume} = \int_0^{2\pi}\int_0^{3/\sqrt{5}}\int_{2r}^{\sqrt{9-r^2}} r\,dz\,dr\,d\theta =$$

$r=0 \Rightarrow u = 9$

$r = 3/\sqrt{5} \Rightarrow u = 9 - \frac{9}{5} = \frac{36}{5}$

0,5

$$= \int_0^{2\pi}\int_0^{3/\sqrt{5}} \left(\sqrt{9-r^2}\,r - 2r^2\right) dr\,d\theta =$$

$\begin{cases} u = 9 - r^2 \\ du = -2r\,dr \end{cases}$

$$= 2\pi \int_0^{3/\sqrt{5}} \sqrt{9-r^2}\,r\,dr - 2\pi\int_0^{3/\sqrt{5}} 2r^2\,dr$$

0,5

$$= \cancel{2}\pi \int_9^{36/5} \frac{1}{-\cancel{2}}\,u^{1/2}\,du - \left(4\pi\frac{r^3}{3}\Big|_0^{3/\sqrt{5}}\right)$$

$$= \pi\,\frac{u^{3/2}}{3/2}\Big|_{36/5}^{9} - \frac{4\pi}{3}\left(3/\sqrt{5}\right)^3 = \frac{2\pi}{3}\left[9^{3/2} - \left(\frac{36}{5}\right)^{3/2}\right] - \frac{4\pi}{3}\left(3/\sqrt{5}\right)^3$$

0,5

$$= 18\pi\left(1 - \frac{2}{\sqrt{5}}\right)$$

$$\text{Volume} = \iint_R \left(\int_{2\sqrt{x^2+y^2}}^{\sqrt{9-x^2-y^2}} dz\right) dx\,dy = \int_{-\frac{3}{\sqrt{5}}}^{\frac{3}{\sqrt{5}}} \int_{-\sqrt{\frac{9}{5}-y^2}}^{\sqrt{\frac{9}{5}-y^2}} \left(\sqrt{9-x^2-y^2} - 2\sqrt{x^2+y^2}\right) dx\,dy$$

3) <u>outra resolução</u> (coordenadas esféricas) $\therefore$ $x = \rho \operatorname{sen}\varphi \cos\theta$, $y = \rho \operatorname{sen}\varphi \operatorname{sen}\theta$

$$z = \rho\cos\varphi, \quad \rho = \sqrt{x^2+y^2+z^2}$$

$$\begin{cases} x^2+y^2+z^2 = 9 \\ z = 2\sqrt{x^2+y^2} \Rightarrow z^2 = 4(x^2+y^2) \end{cases} \Rightarrow 9 = 5(x^2+y^2) \Rightarrow x^2+y^2 = \frac{9}{5} = \left(\frac{3}{\sqrt{5}}\right)^2$$

$$\begin{cases} z = 2\cdot\frac{3}{\sqrt{5}} = \frac{6}{\sqrt{5}} \\ \rho = 3 \end{cases}$$

$$\begin{cases} z = \frac{6}{\sqrt{5}} = \rho\cos\varphi_0 = 3\cos\varphi_0 \\ z = \frac{6}{\sqrt{5}} = 2\rho\operatorname{sen}\varphi_0 = 6\operatorname{sen}\varphi_0 \end{cases}$$

$$\operatorname{tg}\varphi_0 = \frac{1}{2}, \quad \cos\varphi_0 = \frac{2}{\sqrt{5}}, \quad \operatorname{sen}\varphi_0 = \frac{1}{\sqrt{5}}$$

$$B: \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq \rho \leq 3 \\ 0 \leq \varphi \leq \varphi_0 \end{cases}$$

1,2

$$V = \int_0^{2\pi}\int_0^3\int_0^{\varphi_0} \rho^2 \operatorname{sen}\varphi \, d\varphi\, d\rho\, d\theta = 2\pi\left(\int_0^3 \rho^2\, d\rho\right)\left(\int_0^{\varphi_0}\operatorname{sen}\varphi\, d\varphi\right)$$

0,8

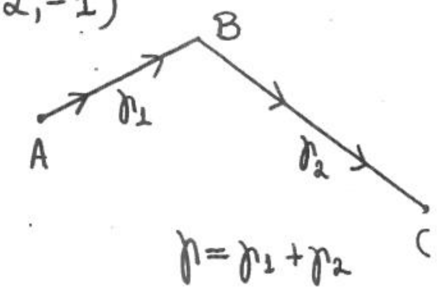
$$= 2\pi\left(\frac{\rho^3}{3}\Big|_0^3\right)\left(-\cos\varphi\Big|_0^{\varphi_0}\right) = 2\pi\,\frac{3^3}{3}\left(1-\cos\varphi_0\right)$$

$$= 2\pi\cdot 9\left(1-\frac{2}{\sqrt{5}}\right) = 18\pi\left(1-\frac{2}{\sqrt{5}}\right)$$

0,5

4) $A=(0,1,-1)$, $B=(1,1,0)$, $C=(1,2,-1)$

$\vec{F}(x,y,z)=(y^2, x, -z)$



$\vec{AB}=B-A=(1,0,1)$, $\vec{BC}=C-B=(0,1,-1)$

$\gamma_1(t)=A+t\vec{AB}=(0,1,-1)+t(1,0,1)$

$$\boxed{\gamma_1(t)=(t,1,-1+t),\quad 0\leq t\leq 1}$$

0,4

$\gamma_2(t)=B+t\vec{BC}=(1,1,0)+t(0,1,-1)$

$$\boxed{\gamma_2(t)=(1,1+t,-t),\quad 0\leq t\leq 1}$$

0,4

$\vec{F}(\gamma_1(t))=(1,t,1-t)$, $\gamma_1'(t)=(1,0,1)$

$\vec{F}(\gamma_1(t))\cdot\gamma_1'(t)=1+0+1-t=2-t$

0,4

$\vec{F}(\gamma_2(t))=((1+t)^2,1,t)$, $\gamma_2'(t)=(0,1,-1)$

$\vec{F}(\gamma_2(t))\cdot\gamma_2'(t)=1-t$

0,4

$$\text{TRABALHO}=\int_\gamma \vec{F}\cdot d\gamma=\int_{\gamma_1}\vec{F}\cdot d\gamma_1+\int_{\gamma_2}\vec{F}\cdot d\gamma_2$$

$$=\int_0^1 \vec{F}(\gamma_1(t))\cdot\gamma_1'(t)\,dt+\int_0^1\vec{F}(\gamma_2(t))\cdot\gamma_2'(t)\,dt$$

0,7

$$=\int_0^1(2-t)\,dt+\int_0^1(1-t)\,dt$$

$$=3-2\int_0^1 t\,dt=2$$

0,2